

BPS domain walls in $\mathcal{N} = 4$ supergravity and dual flows

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Abstract

We establish the conditions for supersymmetric domain wall solutions to $\mathcal{N} = 4$ gauged supergravity in five dimensions. These read as BPS first-order equations for the warp factor and the scalar fields, driven by a superpotential and supplemented by a set of constraints that we specify in detail. Then we apply our results to certain consistent truncations of IIB supergravity, thus exploring their dual field theory renormalization group flows. We find a universal flow deforming superconformal theories on D3-branes at Calabi–Yau cones. Moreover, we obtain a superpotential for the solution corresponding to the baryonic branch of the Klebanov–Strassler theory, as well as the superpotential for the flow describing D3 and wrapped D5-branes on the resolved conifold.

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1 Introduction

According to the gauge/gravity correspondence, domain wall solutions in (super)gravity theories describe dual renormalization group (RG) flows in which the extra radial coordinate plays the role of the energy scale. Some important early examples for the development of these concepts include [1, 2]. In these references, the domain wall interpolates between two AdS solutions and is dual to an RG flow between two different conformal field theories. More sophisticated flows, displaying remarkable properties like confinement and chiral symmetry breaking, can be constructed based on the conifold geometry [3, 4, 5, 6, 7].

In this paper we will develop a systematic approach to this important class of solutions in five-dimensional, $\mathcal{N} = 4$ (i.e. half-maximal) gauged supergravity. For our purposes, a (flat) domain wall will be a gravitational solution that preserves four-dimensional Poincaré invariance. The general form of the metric thus reads

$$ds^2 = e^{2A(r)} ds^2(\mathbb{R}^{1,3}) + dr^2 \quad (1.1)$$

where A is a warp factor, function of the radial coordinate r . These metrics are typically supported by a set of scalar fields ϕ^x with a radial profile. An important subclass of domain walls is obtained when the scalar potential can be written, in a very specific way, in terms of a real function, $W(\phi)$, known as the superpotential. In this case, the second-order equations of motion can be solved by imposing a BPS-like condition, which ensures gravitational stability and reads as first-order equations for the radial evolution of the domain wall [8]:

$$A' = W, \quad \phi^{x'} = -3 g^{xy} \partial_y W, \quad (1.2)$$

where g_{xy} is the scalar kinetic matrix. The superpotential itself is a noteworthy quantity: it defines the tension of the domain wall and provides a monotonic c -function for the dual RG flow. The conditions above emerge naturally when supersymmetry is present: the superpotential and the first-order flow equations arise from the vanishing of the fermionic susy variations. Supergravity provides thus a natural setup to look for them [9]. Nevertheless, given the scalar potential, a superpotential W reproducing it is not necessarily unique nor related a priori with supersymmetry.

An analysis of supersymmetric domain walls in five-dimensional $\mathcal{N} = 2$ supergravity was done some time ago in [10, 11]. The results of [11] were recently applied in [12] to study domain walls in the $\mathcal{N} = 2$ truncations of [13], lifting to type IIB solutions based on the conifold geometry. Nonetheless, the various conifold solutions are contained in different $\mathcal{N} = 2$ truncations, and the one describing the baryonic branch of the Klebanov–Strassler theory, found in [7] building on a proposal of [14], falls outside the setup of [12]. However, a larger consistent truncation has been established that preserves $\mathcal{N} = 4$ supersymmetry and includes all known conifold solutions [13, 15]. This $\mathcal{N} = 4$ theory thus provides the proper framework for a unified, systematic study of the supersymmetry properties of those solutions. Working in this setup, besides giving a genuine supergravity origin to the known superpotentials (which were derived by other means), we will provide a superpotential for the solution of [7], clarifying an issue recently put forward in the literature [12, 16].

The conifold solutions preserve four supercharges, and as we will see 1/4 BPS domain walls emerge naturally in $\mathcal{N} = 4$ gauged supergravity. Our general analysis is inspired by [11], but is considerably more involved, since in our case the R-symmetry is $\text{USp}(4)$ rather than

just $SU(2)$, and three-quarters of the supersymmetry is broken. Some results on $1/2$ BPS domain walls in $\mathcal{N} = 4$ appeared in [17].

The paper is divided in two distinct parts. In the first we analyze the general construction of supersymmetric domain walls in five-dimensional, $\mathcal{N} = 4$ gauged supergravity. As we summarize in section 3.2, we find that the necessary and sufficient conditions for supersymmetry to be preserved take the form of first-order flow equations for the warp factor and the scalar fields, generated by a superpotential computed as an eigenvalue of the gravitino shift matrix. In addition, we obtain a set of algebraic constraints involving the scalars. Moreover, the $\mathcal{N} = 4$ supersymmetry parameter is constrained in such a way that four supercharges are generically preserved.

In the second part, we focus on two distinguished $\mathcal{N} = 4$ supergravity models. These are significant because they can be obtained as consistent truncations of type IIB supergravity, meaning that any solution to the five-dimensional theory is automatically a solution to the ten-dimensional one. The first model arises from dimensional reduction on any manifold admitting a Sasaki–Einstein structure as described in [18, 19]. By implementing our BPS conditions, we provide its most general supersymmetric domain wall solution. This describes universal deformations of the superconformal field theories on D3-branes at the tip of Calabi–Yau cones. Tuning a parameter corresponding in type IIB to an imaginary self-dual three-form flux, this new flow interpolates between the solution in [20, 21] and the one in [22].

The second model is an extension of the preceding one when the base of the conifold is chosen as internal Sasaki–Einstein geometry. Its coset structure permits an $SU(2) \times SU(2)$ left-invariant reduction that enlarges the consistent truncation to include a new vector multiplet along with new gaugings [13, 15]. One can see this truncation as an $\mathcal{N} = 4$ supersymmetrization of the Papadopoulos–Tseytlin (PT) ansatz [23], which encompasses all the celebrated conifold solutions. Again, by implementing the general conditions found in the first part of the paper, and exploiting the $\mathcal{N} = 4$ structure identified in [13], we study the domain walls within the PT subsector of the model. The various conifold solutions arise by solving the algebraic constraints in diverse manners, and their previously known superpotentials, obtained with different procedures, are given an explicit supergravity origin in this context. In this way, we extract the superpotential that generates the flow describing the baryonic branch of the Klebanov–Strassler theory, which was previously unknown. Furthermore, we obtain the superpotential for a supersymmetric solution describing D3 and wrapped D5-branes on the resolved conifold, that differs from the non-supersymmetric one in [6].

The paper is organized as follows. In section 2 we present the theory under study, i.e. five-dimensional, $\mathcal{N} = 4$ gauged supergravity. The main results regarding the general

analysis of BPS domain walls are stated in section 3, where we also present the analysis of the supersymmetry equations from the gravity multiplet. In sections 4 and 5 we apply our general conditions to the consistent truncations of type IIB string theory and construct their superpotentials and domain wall solutions. We conclude in section 6. Some technical details on the $\mathcal{N} = 4$ scalar manifold are given in appendix A, while appendix B contains the comprehensive analysis of the gaugino equation.

2 $\mathcal{N} = 4$ gauged supergravity in five dimensions

The ungauged $\mathcal{N} = 4$ supergravity theory in five dimensions was constructed in [24], together with its $SU(2)$ -gauged version. The generally gauged theory was given in [25], and further extended in [26] by using the embedding tensor formalism (see e.g. [27] for a review). In the following, we provide a compendium of the features that will be relevant. Since we will study domain wall solutions in which only the metric and the scalar fields acquire a non-trivial profile, it will be sufficient to discuss the couplings of the scalars and their appearance in the supersymmetry transformations of the fermions.

2.1 Geometry of the scalar manifold

The scalars of five-dimensional $\mathcal{N} = 4$ supergravity coupled to n vector multiplets define a σ -model whose target manifold is

$$\mathcal{M}_{\text{scalars}} = SO(1, 1) \times \frac{SO(5, n)}{SO(5) \times SO(n)}. \quad (2.1)$$

The $SO(1, 1)$ factor is parameterized by a real scalar σ , being part of the gravity multiplet, while the second factor is parameterized by real scalars ϕ^x , $x = 1, \dots, 5n$, being part of the vector multiplets. The $\frac{SO(5, n)}{SO(5) \times SO(n)}$ coset representative is $(\mathcal{V}_M^{\underline{a}}, \mathcal{V}_M^a)$, where $M = 1, \dots, 5+n$ labels the standard representation of $SO(5, n)$, while $\underline{a} = 1, \dots, 5$ and $a = 1, \dots, n$ label the representations of $SO(5)$ and $SO(n)$, respectively. So we have that the global $SO(5, n)$ acts from the left, while the local $SO(5) \times SO(n)$ acts from the right.

The coset manifold is equipped with an invariant metric and an $SO(5) \times SO(n)$ composite connection. These are encoded in the one-form $\mathcal{V}^{-1}d\mathcal{V}$, which takes values in the $\mathfrak{so}(5, n)$ algebra and can be expressed as

$$\mathcal{V}^{-1}d\mathcal{V} = 2v^{ab}t_{\underline{a}\underline{b}} - \omega^{ab}t_{ab} + \omega^{\underline{a}\underline{b}}t_{\underline{a}\underline{b}}, \quad (2.2)$$

where we are splitting the t_{MN} generators of $\mathfrak{so}(5, n)$ into $\mathfrak{so}(5)$ generators $t_{\underline{a}\underline{b}}$, $\mathfrak{so}(n)$ generators t_{ab} and coset generators $t_{\underline{a}b}$. For definiteness, we work in the fundamental representation

$(t_{MN})_P{}^Q = \delta_{[M}^Q \eta_{N]P}$, where $\eta = \text{diag}\{- - - - + \dots +\}$ is the $\text{SO}(5, n)$ metric. In our expansion above,

$$v^{ab} \equiv v_x^{aa} d\phi^x = -(\mathcal{V}^{-1} d\mathcal{V})^{ab} \quad (2.3)$$

are the coset vielbeine, while

$$\omega^{ab} = (\mathcal{V}^{-1} d\mathcal{V})^{ab}, \quad \omega^{ab} = (\mathcal{V}^{-1} d\mathcal{V})^{ab} \quad (2.4)$$

are the $\text{SO}(5)$ and $\text{SO}(n)$ pieces of the connection, respectively. The vielbein defines an invariant metric g_{xy} on the coset via

$$v_x^{aa} v_y^{aa} = g_{xy}, \quad (2.5)$$

and satisfies

$$v_x^{aa} v^{x bb} = \delta^{ab} \delta^{ab}. \quad (2.6)$$

The R-symmetry group of $\mathcal{N} = 4$ supergravity in five dimensions is $\text{Spin}(5) \simeq \text{USp}(4)$. We denote by $i, j = 1, \dots, 4$ its spinorial representation, in which the fermion fields as well as the supersymmetry parameters transform. In order to switch from spinor to vector representation, we use the Clifford map defined by the $\text{Cliff}(5)$ Dirac matrices $(\Gamma^a)_i{}^j$. This acts on an antisymmetric p -tensor $T_{\underline{a_1 \dots a_p}}$ as

$$T_{\underline{a_1 \dots a_p}} \mapsto T_{ij} = T_{\underline{a_1 \dots a_p}} \Gamma^{\underline{a_1 \dots a_p}}_{ij}, \quad (2.7)$$

where $\Gamma^{\underline{a_1 \dots a_p}}$ is the antisymmetric product of p Γ -matrices. For instance, for the vielbein we have¹

$$v^{a ij} = v^{aa} \Gamma_a^{ij}. \quad (2.8)$$

Using the properties of the Γ -matrices summarized in appendix A, the relations (2.5), (2.6) can equivalently be expressed as

$$v_x^{a ij} v_y^a{}_{ij} = 4 g_{xy}, \quad v_x^{a ij} v^{x b}{}_{kl} = \delta^{ab} (4 \delta_{kl}^{ij} - \Omega^{ij} \Omega_{kl}). \quad (2.9)$$

Here, Ω is the $\text{USp}(4)$ invariant tensor, satisfying

$$\Omega_{ij} = -\Omega_{ji}, \quad \Omega_{ij} = \Omega^{ij}, \quad \Omega_{ij} \Omega^{jk} = -\delta_i^k. \quad (2.10)$$

¹We will have one exception to (2.7): in order to match the expressions in [24], we introduce a factor of $1/2$ in the Clifford map for the coset representative, i.e. we define

$$\mathcal{V}_M i^j = \frac{1}{2} \mathcal{V}_M^a \Gamma_a i^j.$$

This is used to raise and lower the indices, according to the NW-SE rule

$$V^i = \Omega^{ij} V_j, \quad V_i = V^j \Omega_{ji}. \quad (2.11)$$

We will often omit the $\text{USp}(4)$ spinorial indices i, j . It is understood that $\text{USp}(4)$ matrices S, T are multiplied according to the rule $S_i^k T_k^j$. On the other hand, we will always explicitly display the vectorial indices $\underline{a}, \underline{b}$. The $\underline{a}, \underline{b}$ (as well as the a, b) indices are always raised and lowered with the Kronecker delta. Further useful relations satisfied by the quantities defined above are given in appendix A.

2.2 Supersymmetry transformations

Once the number of vector multiplets is fixed, the $\mathcal{N} = 4$ gauged supergravity theory is fully characterized by the embedding tensor, which is a constant matrix specifying how the gauge group is embedded into the global symmetry group $\text{SO}(1, 1) \times \text{SO}(5, n)$. The embedding tensor assigns charges to the different fields, and appears in various terms of the gauged supergravity action (gauge covariant derivatives, fermion mass terms, scalar potential) as well as in the supersymmetry transformations, where the extra terms due to the gauging are known as fermionic shifts. As in [26], we denote its components by $f^{MNP} = f^{[MNP]}$, $\xi^{MN} = \xi^{[MN]}$ and ξ^M .

In the following we provide the supersymmetry transformations of the fermions, which will be our starting point for the analysis of supersymmetric domain walls. The $\mathcal{N} = 4$ supergravity fermions are the gravitini $\psi_{\mu i}$, the spin 1/2 fermions χ_i , both sitting in the gravity multiplet, and the gaugini λ_i^a in the vector multiplets. Note that they all carry a $\text{USp}(4)$ index. Moreover, they are all symplectic-Majorana. We will restrict to the case in which all the one- and two-form fields in the theory, as well as the ξ^M components of the embedding tensor, vanish. Then the variations given in [26] read

$$\delta\psi_{\mu i} = D_\mu \epsilon_i - \frac{i}{2} g P_i^j \gamma_\mu \epsilon_j, \quad (2.12)$$

$$\delta\chi_i = -\frac{i}{2} \gamma^\mu \partial_\mu \sigma \epsilon_i + \frac{3}{2} g \partial_\sigma P_i^j \epsilon_j, \quad (2.13)$$

$$\delta\lambda_i^a = -\frac{i}{2} \gamma^\mu \partial_\mu \phi^x v_{x ij}^a \epsilon^j - g P_i^a{}^j \epsilon_j, \quad (2.14)$$

where the $\mathcal{N} = 4$ supersymmetry parameter ϵ_i is also a $\text{USp}(4)$ symplectic-Majorana spinor, γ^μ are the $\text{Cliff}(1, 4)$ Dirac matrices, g is the gauge coupling constant (assumed positive), and we introduced

$$\Sigma = e^{\sigma/\sqrt{3}}. \quad (2.15)$$

The derivative D_μ is covariant both with respect to the $\text{SO}(1, 4)$ Lorentz and the $\text{USp}(4)$ R-symmetry transformations on the spinor ϵ_i . Moreover, we defined the gravitino shift matrix

$$P_{ij} = P^{ab} \Gamma_{\underline{ab}ij}, \quad (2.16)$$

with

$$P^{ab} := -\frac{1}{6\sqrt{2}} \Sigma^2 \xi^{ab} + \frac{1}{36} \Sigma^{-1} \epsilon^{abcde} f^{cde}, \quad (2.17)$$

where we are using the “dressed” components of the embedding tensor

$$\xi^{ab} := \xi^{MN} \mathcal{V}_M^a \mathcal{V}_N^b, \quad f^{abc} := f^{MNP} \mathcal{V}_M^a \mathcal{V}_N^b \mathcal{V}_P^c. \quad (2.18)$$

The $\text{USp}(4)$ matrix P_{ij} is going to be our “matrix superpotential”, out of which we will extract the actual superpotential generating the flow equations. We have also introduced the gaugino shift matrix

$$P_{ij}^a := \frac{1}{\sqrt{2}} \Sigma^2 \xi^{MN} \mathcal{V}_M^a \mathcal{V}_{Nij} + \Sigma^{-1} f^{MNP} \mathcal{V}_M^a \mathcal{V}_{Ni}{}^k \mathcal{V}_{Pkj}. \quad (2.19)$$

This can only in part be written as a derivative of the gravitino shift (see appendix B).

3 Supersymmetric domain walls

In this section, we come to study the conditions for supersymmetric domain walls.

3.1 General strategy and preliminary decomposition

We specialize to a (flat) domain wall ansatz, namely we take a spacetime metric of the form (1.1) and we assume that all scalars as well as the supersymmetry parameter ϵ_i just depend on the fifth, radial coordinate r . We also switch off all one- and two-forms in the theory.

The supersymmetry conditions follow from setting to zero the fermion variations given in the previous section. In detail, the $\mathbb{R}^{1,3}$ components of the gravitino variation (2.12) yield the condition

$$-A' \gamma_5 \epsilon_i + g i P_i^j \epsilon_j = 0, \quad (3.1)$$

while the radial component imposes

$$\epsilon'_i + \phi^{x'} \omega_{xi}{}^j \epsilon_j - \frac{i}{2} g P_i^j \gamma_5 \epsilon_j = 0, \quad (3.2)$$

where we recall that $\omega_{xi}{}^j$ is the $\text{SO}(5)$ connection on the scalar manifold. The variation (2.13) of the other fermion in the gravity multiplet yields

$$\sigma' \gamma_5 \epsilon_i + 3ig \partial_\sigma P_i^j \epsilon_j = 0, \quad (3.3)$$

and finally the gaugino variation (2.14) gives

$$i\phi^{x'} v_{xij}^a \gamma_5 \epsilon^j + 2g P_i^a{}^j \epsilon_j = 0. \quad (3.4)$$

A special role in our analysis will be played by the gravitino shift matrix (2.16), (2.17). As we will show, fixing it generically reduces the R-symmetry of the solution from $\text{USp}(4)$ to $\text{U}(1)$, namely it produces 1/4 BPS configurations. Furthermore, its eigenvalues correspond to the possible superpotentials generating the flow equations. In the following, we split the gravitino matrix in a way that is most suitable for making these features manifest.

Let us first describe how a $\text{USp}(4) \rightarrow \text{SU}(2)$ breaking is defined. The gravitino shift matrix can be used to construct an $\text{SO}(5)$ vector

$$\tilde{X}^a = \epsilon^{abcde} P_{bc} P_{de}, \quad (3.5)$$

whose norm is

$$|\tilde{X}| \equiv \sqrt{\tilde{X}^a \tilde{X}_a} = \sqrt{8 (P^{ab} P_{ab})^2 - 16 P^{ab} P_{bc} P^{cd} P_{da}}. \quad (3.6)$$

Away from the zero locus, we can introduce a normalized vector

$$X = \tilde{X} / |\tilde{X}|. \quad (3.7)$$

which pointwise on the scalar manifold specifies a preferred direction in \mathbb{R}^5 and therefore an $\text{SO}(4)$ subgroup of the R-symmetry group $\text{SO}(5)$. On the spinors, this defines a reduction $\text{USp}(4) \rightarrow \text{SU}(2)_+ \times \text{SU}(2)_-$, where the plus and minus refer to the chiralities defined by $X_i^j = X_a \Gamma_a^{ij}$. The projectors over these $\text{SU}(2)_\pm$ are

$$\Pi_{\pm i}{}^j = \frac{1}{2} (\delta_i^j \pm X_i^j) \quad (3.8)$$

(note that the projector condition $\Pi_\pm^2 = \Pi_\pm$ is satisfied since $X_i^k X_k^j = \delta_i^j$). As we will see, on supersymmetric domain walls either one of these projectors acts on the $\text{Usp}(4)$ supersymmetry parameter, constraining it to be an $\text{SU}(2)$ spinor.

The matrix P actually constrains the supersymmetry parameter even further. We note that P^{ab} lives in the $\text{so}(4)$ identified by X , since it is satisfied that

$$X_a P^{ab} = 0, \quad (3.9)$$

which follows from the definition of X together with the identity $\epsilon^{abcde}T_{ab}T_{cd}T_{ef} = 0$, holding for any antisymmetric tensor T_{ab} in five dimensions. Note that (3.9) can also be written as

$$X P = P X. \quad (3.10)$$

Hence the P matrix can be decomposed as the sum of two matrices in $\mathfrak{so}(4) = \mathfrak{su}(2)_+ \oplus \mathfrak{su}(2)_-$:

$$P = P_+ + P_-, \quad (3.11)$$

with the $\mathfrak{su}(2)_\pm$ components being

$$P_\pm := P \Pi_\pm = \Pi_\pm P. \quad (3.12)$$

Squaring $P = P_{ab}\Gamma^{ab}$, we find²

$$P^2 = -2P_{ab}P^{ab}1_4 + \tilde{X}. \quad (3.13)$$

Noting that $P_+P_- = 0$, we deduce

$$P_\pm^2 = -W_\pm^2 \Pi_\pm, \quad (3.14)$$

where we defined the real functions of the scalar fields

$$\begin{aligned} W_\pm &= \sqrt{2 P^{ab}P_{ab} \mp |\tilde{X}|} \\ &= \sqrt{2 P^{ab}P_{ab} \mp \sqrt{8 (P^{ab}P_{ab})^2 - 16 P^{ab}P_{bc}P^{cd}P_{da}}}. \end{aligned} \quad (3.15)$$

One can see that $+W_\pm$ and $-W_\pm$ coincide with the four real eigenvalues of the hermitian matrix iP . Either W_+ or W_- will play the role of the superpotential generating the flow equations. For $W_\pm \neq 0$, we can introduce normalized matrices

$$Q_\pm = \frac{i}{W_\pm} P_\pm \quad (3.16)$$

which satisfy

$$Q_\pm^2 = \Pi_\pm \quad (3.17)$$

as well as

$$X Q_\pm = Q_\pm X = \pm Q_\pm. \quad (3.18)$$

²The fact that the square of the gravitino shift matrix P is not just proportional to the identity is a crucial difference with respect to the $\mathcal{N} = 2$ analysis of [11]. Of course, this is because the former is a $\text{USp}(4)$ matrix while the latter is an $\text{SU}(2)$ matrix. This also motivates why in the $\mathcal{N} = 4$ case we generically obtain 1/4 BPS domain walls while in the $\mathcal{N} = 2$ case 1/2 BPS configurations are found.

So we have

$$P = -i W_+ Q_+ - i W_- Q_- . \quad (3.19)$$

As we will discuss below, the Q matrix will be involved in an algebraic constraint acting on the $\text{USp}(4)$ spinor parameter, breaking the R-symmetry of the solution down to $\text{U}(1)$.

It will be useful to also introduce generators $(L_\pm^r)_i{}^j$, $r = 1, 2, 3$, for the $\text{su}(2)_\pm$ subalgebras of $\text{so}(5)$ identified by X . These verify³

$$X L_\pm^r = L_\pm^r X = \pm L_\pm^r , \quad (3.21)$$

$$L_\pm^r L_\pm^s = -\delta^{rs} \Pi_\pm + \epsilon^{rst} L_\pm^t , \quad (3.22)$$

$$L_+^r L_-^s = 0 , \quad (3.23)$$

together with the completeness relation

$$L_{\pm ij}^r L_{\pm}^{r kl} = 2 \Pi_{\pm i}^{(k} \Pi_{\pm j}^{l)} . \quad (3.24)$$

So our P can be written as

$$P = P_+^r L_+^r + P_-^r L_-^r = -i W_+ Q_+^r L_+^r - i W_- Q_-^r L_-^r . \quad (3.25)$$

Having split the gravitino shift matrix in a suitable way and having introduced the main characters playing a role in our analysis, we can now present our results.

3.2 The necessary and sufficient conditions

Our main result in this first part of the paper is that the equations (3.1)–(3.4) for a supersymmetric domain wall in $\mathcal{N} = 4$ gauged supergravity are equivalent to a system of first-order flow equations for the scalar fields, generated by the superpotential W_+ , or W_- , introduced in eq. (3.15) above, and supplemented by a set of constraints. Specifically, if W_+ is chosen,

³ An explicit realization of these $\text{su}(2)_\pm$ generators is the following. First we note that the unit $\text{SO}(5)$ vector X can be put along e.g. the fifth direction by exploiting the local $\text{SO}(5)$ symmetry transformations which act on the $\text{SO}(5, n)/(\text{SO}(5) \times \text{SO}(n))$ coset representative from the right. This could in principle be done only at a fixed value of the $\text{SO}(1,1)$ scalar σ , because a priori X depends on it. However, in our analysis we will find that on supersymmetric domain walls, X has to be independent of σ , see eq. (3.29) below. Hence we can take $X_{\underline{a}} = \delta_{\underline{a}}^5$, or in spinorial indices $X_i{}^j = \Gamma_5{}_i{}^j$. Then the $\text{su}(2)_\pm$ generators can be taken as:

$$L_\pm^1 = \frac{1}{2} (\pm \Gamma_{12} - \Gamma_{34}) , \quad L_\pm^2 = \frac{1}{2} (\pm \Gamma_{13} + \Gamma_{24}) , \quad L_\pm^3 = \frac{1}{2} (\pm \Gamma_{14} - \Gamma_{23}) . \quad (3.20)$$

we find the conditions

$$A' = \pm g W_+, \quad (3.26)$$

$$\sigma' = \mp 3g \partial_\sigma W_+, \quad (3.27)$$

$$\phi^{x'} = \mp 3g g^{xy} \partial_y W_+, \quad (3.28)$$

together with

$$\partial_\sigma X = 0 = \partial_\sigma Q_+^r, \quad (3.29)$$

$$\phi^{x'} D_x X = 0 = \phi^{x'} D_x Q_+^r, \quad (3.30)$$

(which can equivalently be written as $\partial_\sigma Q_+ = 0$ and $\phi^{x'} D_x Q_+ = 0$, respectively) and

$$\xi^{a\bar{a}} X_{\bar{a}} = 0, \quad (3.31)$$

$$f^{a\bar{a}b} L_{+\bar{a}b}^r (\delta^{rs} + Q_+^r Q_+^s) = 0. \quad (3.32)$$

In the last two relations we used the dressed components of the embedding tensor

$$\xi^{a\bar{a}} := \xi^{MN} \mathcal{V}_M^a \mathcal{V}_N^{\bar{a}}, \quad f^{a\bar{a}b} := f^{MNP} \mathcal{V}_M^a \mathcal{V}_N^{\bar{a}} \mathcal{V}_P^b. \quad (3.33)$$

Moreover, the $\mathcal{N} = 4$ supersymmetry parameters are constrained by

$$Q_{+i}{}^j \epsilon_j = \pm \gamma_5 \epsilon_i, \quad (3.34)$$

which means that the domain wall preserves four out of sixteen supercharges, i.e. it preserves $\mathcal{N} = 1$ supersymmetry. This can be seen by first noting that by eq. (3.17) our constraint implies $X_i{}^j \epsilon_j = \epsilon_i$. This means that the supersymmetry parameter ϵ_i , which generically is a $\text{USp}(4)$ symplectic-Majorana spinor carrying sixteen real degrees of freedom, is actually restricted to be an $\text{SU}(2)$ symplectic-Majorana spinor, carrying eight real degrees of freedom. Then one observes that on such spinors Q_+ squares to the identity and therefore (3.34) defines a projector halving the number of degrees of freedom, so that we are left with four independent supercharges.

As we will discuss below, these supersymmetry conditions imply the equations of motion of $\mathcal{N} = 4$ supergravity and we therefore have a full solution of the five-dimensional theory.

A few comments on the constraints. The first condition (3.29) tells us that the choice of $\text{so}(4) \subset \text{so}(5)$ does not depend on σ , while the second one constrains the projector related to the domain wall direction.⁴ Conditions (3.30) tell that X and Q are constant with respect to

⁴The latter is analogous to the condition $\partial_x Q^r = 0$ found in the $\mathcal{N} = 2$ analysis of [11], while the former was not present in that case.

the $\text{SO}(5)$ covariant derivative evaluated along the flow. Finally, for each value of the index a , namely for each vector multiplet, eq. (3.31) states that the $\text{so}(5)$ vector $\xi^{a\bar{a}}$ is orthogonal to $X_{\bar{a}}$, while (3.32) tells that the $\text{su}(2)_+$ piece of the $\text{so}(5)$ algebra valued matrix $f^{a\bar{a}b}$ has to be along Q .

If W_- is chosen instead of W_+ , one has the same conditions (3.26)–(3.34), with the only change that W_+, Q_+, L_+^r are everywhere replaced by W_-, Q_-, L_-^r , respectively. Notice that in this case $X_i^j \epsilon_j = -\epsilon_i$.

The flow equation (3.26) for the warp factor, the one (3.27) for the scalar σ , and the constraints (3.29), (3.30), (3.34), arise from the supersymmetry variation of the fermions in the $\mathcal{N} = 4$ gravity multiplet. This is proved in the next section. The other conditions, namely the flow equations for the remaining scalar fields (3.28) and constraints (3.31), (3.32), come from the variation of the gaugini in the vector multiplets. The complete proof of this statement is very non-trivial, and represents a major piece of our analysis; however since it involves many technical steps we present it in appendix B. There, we also show that the second in (3.30) is implied by the gaugino equation (see eq. (B.28)), so it does not need to be checked independently.

Finally, we remark that the results above assume $|\tilde{X}| \neq 0$. In the special case $|\tilde{X}| = 0$, which implies $\tilde{X} = 0$, the projection Π_{\pm} on $\text{SU}(2)_{\pm}$ cannot be defined and the analysis proceeds in a slightly different way. Though in this case we have not performed a complete study of the constraints arising from the gaugino equation, one can see that the first-order flow equations for the warp factor and the scalar fields still hold, but at the place of W_{\pm} there is just one possible superpotential, $W = \sqrt{2P_{\underline{ab}}P^{\underline{ab}}}$, and the gravitino shift matrix can be written as $P = -iWQ$, with $Q^2 = 1_4$. The only constraint on the supersymmetry parameter from the gravitino equation is $Q_i^j \epsilon_j = \gamma_5 \epsilon_i$, which is a projection halving the amount of supersymmetry. So in this $|\tilde{X}| = 0$ case we expect 1/2 BPS domain walls, preserving eight Poincaré supercharges.

3.3 Analysis of the equations from the gravity multiplet

In this section, we prove that eqs. (3.1)–(3.3), ensuring that the variation of the fermions in the gravity multiplet vanishes, are equivalent to eqs. (3.26), (3.27), (3.29), (3.30) and (3.34) when W_+ is chosen. The proof for W_- works in the same way.

We start from the gravitino equation (3.1), i.e.

$$g i P_i^j \epsilon_j = A' \gamma_5 \epsilon_i. \quad (3.35)$$

Applying P to this equation and using (3.13), we obtain

$$-g^2 \tilde{X}_i^j \epsilon_j = \left[(A')^2 - 2g^2 P_{\underline{ab}} P^{\underline{ab}} \right] \epsilon_i. \quad (3.36)$$

Applying \tilde{X} to this equation, one immediately sees that its solution is the flow equation (3.26) for the warp factor A :

$$A' = \pm g W_+, \quad (3.37)$$

with W_+ as in (3.15), and with ϵ_i satisfying

$$X_i^j \epsilon_j = \epsilon_i. \quad (3.38)$$

The latter condition constrains the $\text{USp}(4)$ spinor ϵ_i to be a spinor of $\text{SU}(2)_+$. Note that if $|\tilde{X}| = 0$ this condition does not hold. In the following, we will assume $|\tilde{X}| \neq 0$. Also note that we could equally well have taken W_- in (3.37), and consequently $X_i^j \epsilon_j = -\epsilon_i$, implying that ϵ_i is a spinor of $\text{SU}(2)_-$. In the following we keep on discussing the case in which W_+ is chosen. The argument for the opposite choice is completely analogous and just requires to switch a few signs.

With these assumptions, recalling (3.11), (3.16), and noting that (3.38) implies $Q_{-i}^j \epsilon_j = 0$, in the locus where $W_+ \neq 0$ the original gravitino equation (3.35) becomes just constraint (3.34):

$$Q_{+i}^j \epsilon_j = \pm \gamma_5 \epsilon_i. \quad (3.39)$$

From (3.17) we see that this acts as a projector on the $\text{SU}(2)_+$ spinor, halving its number of degrees of freedom. Also note that this condition actually implies (3.38).

We next consider equation (3.3) arising from the transformation of the fermions χ_i in the gravity multiplet, namely

$$3i g \partial_\sigma P_i^j \epsilon_j = -\sigma' \gamma_5 \epsilon_i. \quad (3.40)$$

Recalling (3.19) and the algebraic constraints on the spinor obtained above, this can be rewritten as

$$3g \partial_\sigma W_+ Q_{+i}^j \epsilon_j + 3g W_+ \partial_\sigma Q_{+i}^j \epsilon_j + 3g W_- \partial_\sigma Q_{-i}^j \epsilon_j = \mp \sigma' Q_{+i}^j \epsilon_j. \quad (3.41)$$

Applying either $(W_+ Q_+ + W_- Q_-)$ or $W_+ \gamma_5$, using again the constraints on the spinor, adding the two resulting expressions and noting that $W_+^2 - W_-^2 = 2|\tilde{X}|$, we arrive at

$$6g W_+ \partial_\sigma W_+ \epsilon_i + 3g |\tilde{X}| \partial_\sigma X_i^j \epsilon_j = \mp 2W_+ \sigma' \epsilon_i. \quad (3.42)$$

We also observe that $X^2 = 1_4$ implies $\partial_\sigma X X = -X \partial_\sigma X$, which applied to ϵ gives

$$\Pi_+ \partial_\sigma X \epsilon = \partial_\sigma X \Pi_- \epsilon = 0. \quad (3.43)$$

Hence projecting eq. (3.42) with Π_+ , for $W_+ \neq 0$ we obtain the flow equation (3.27) for the scalar σ ,

$$\sigma' = \mp 3g \partial_\sigma W_+. \quad (3.44)$$

Then eq. (3.42) becomes $\partial_\sigma X_i^j \epsilon_j = 0$, which is equivalent to

$$\partial_\sigma X = 0. \quad (3.45)$$

This means that the choice of $\mathfrak{so}(4)$ inside $\mathfrak{so}(5)$ does not depend on σ . Taking the derivative of (3.18), it follows that $\partial_\sigma Q_+$ and $\partial_\sigma Q_-$ are in the $\mathfrak{su}(2)_+$ and $\mathfrak{su}(2)_-$ subspaces respectively:

$$X \partial_\sigma Q_\pm = \partial_\sigma Q_\pm X = \pm \partial_\sigma Q_\pm. \quad (3.46)$$

Consequently we deduce that

$$\partial_\sigma Q_- \epsilon = \partial_\sigma Q_- \Pi_- \epsilon = 0. \quad (3.47)$$

Using the conditions just derived, our original equation (3.41) reduces to

$$\partial_\sigma Q_{+i}^j \epsilon_j = 0. \quad (3.48)$$

Since the choice of $\mathfrak{so}(4)$ inside $\mathfrak{so}(5)$ does not depend on σ , our $\mathfrak{su}(2)_\pm$ generators L_\pm^r do not depend on σ either, so we can write $\partial_\sigma Q_+ = \partial_\sigma Q_+^r L_+^r$. Then, applying twice condition (3.48) and using property (3.22) of the $\mathfrak{su}(2)$ generators, we find

$$\partial_\sigma Q_+^r \partial_\sigma Q_+^s L_+^r L_+^s \epsilon = 0 \quad \Rightarrow \quad \partial_\sigma Q_+^r \partial_\sigma Q_+^r = 0 \quad \Rightarrow \quad \partial_\sigma Q_+^r = 0. \quad (3.49)$$

We have thus also obtained both constraints (3.29). Note that they can equivalently be stated as $\partial_\sigma Q_+ = 0$.

Finally, let us consider eq. (3.2) for the radial flow of the spinor,

$$\epsilon'_i + \phi^{x'} \omega_{xi}^j \epsilon_j - \frac{i}{2} g P_i^j \gamma_5 \epsilon_j = 0, \quad (3.50)$$

where we recall that ω_i^j is the $\text{USp}(4)$ connection. It is not hard to see that compatibility with the algebraic condition (3.34) requires

$$\phi^{x'} D_x Q_{+i}^j \epsilon_j = 0. \quad (3.51)$$

Note that acting with $\phi^{x'} D_x$ on (3.17) and using (3.34), (3.51), we obtain $\phi^{x'} D_x X_i^j \epsilon_j = 0$, which is equivalent to

$$\phi^{x'} D_x X = 0, \quad (3.52)$$

namely the $\text{SO}(5)$ vector X has to be covariantly constant along the flow. It follows that the $\mathfrak{so}(4)$ generators L_\pm^r can be chosen to satisfy $\phi^{x'} D_x L_\pm^r = 0$. Then by the same argument used in (3.49), eq. (3.51) reduces to

$$\phi^{x'} D_x Q_+^r = 0. \quad (3.53)$$

We have thus also obtained constraints (3.30). They are equivalent to $\phi^{x'} D_x Q_+ = 0$.

Taking (3.51) into account, and using the flow equation (3.26), the spinor equation becomes

$$(e^{-A/2} \epsilon_i)' + \phi^{x'} \omega_{xi}{}^j (e^{-A/2} \epsilon_j) = 0. \quad (3.54)$$

By performing a local $\text{SO}(5)$ transformation, we could set to zero the component of the $\text{SO}(5)$ connection along the flow, so that the covariant derivatives in (3.52) and (3.53) become partial derivatives, and the solution for the spinor is

$$\epsilon_i = e^{A/2} \tilde{\epsilon}_i, \quad (3.55)$$

where $\tilde{\epsilon}_i$ is a constant spinor satisfying the algebraic constraint (3.34).⁵

This concludes our analysis of the supersymmetry conditions arising from the gravity multiplet. As already mentioned, the full analysis of the gaugino equation is technically more involved, so that we relegated it to appendix B.

3.4 The domain wall effective action

Plugging in the domain wall ansatz, the five-dimensional $\mathcal{N} = 4$ supergravity action (for which we refer to [25, 26]) reduces to the one-dimensional effective action

$$S = \frac{1}{2} \int_{-\infty}^{+\infty} dr e^{4A} [12(A')^2 - (\sigma')^2 - g_{xy} \phi^{x'} \phi^{y'} - 2V] - 4 [e^{4A} A']_{-\infty}^{+\infty} \quad (3.56)$$

whose equations of motion are the same as the Einstein and the scalar equations of five-dimensional supergravity, evaluated in the domain wall ansatz.

Starting from the general formula for the $\mathcal{N} = 4$ scalar potential V written as an algebraic sum of squares of the fermionic shifts (see [26]), with a long computation we found that this can be rewritten as the following algebraic sum of quadratic terms

$$\begin{aligned} g^{-2} V = & \frac{9}{2} g^{xy} \partial_x W \partial_y W + \frac{9}{2} (\partial_\sigma W)^2 - 6 W^2 - 18 W^2 \partial_\sigma Q_{\underline{ab}} \partial_\sigma Q^{\underline{ab}} - 9 |\tilde{X}| (\partial_\sigma X_{\underline{a}})^2 \\ & + \frac{1}{4} \Sigma^4 (\xi^{aa} X_{\underline{a}})^2 + 2 \Sigma^{-2} \text{Tr} [f^{aab} L^r{}_{\underline{ab}} (\delta^{rs} + Q^r Q^s)]^2, \end{aligned} \quad (3.57)$$

⁵If part of the gauge freedom has already been fixed so that the framework of footnote 3 is achieved and X is made constant on the scalar manifold, then eq. (3.52) becomes a *requirement* that the piece of the $\text{SO}(5)$ connection taking values in the orthogonal complement of $\text{so}(4)$ in $\text{so}(5)$ vanishes. The components of the connection appearing in (3.53) and (3.54) can still be set to zero exploiting the residual gauge freedom.

where W, Q, L^r can be either W_+, Q_+, L^r_+ , or W_-, Q_-, L^r_- . It follows that our effective action can be recast in a BPS form as

$$\begin{aligned}
S = & \frac{1}{2} \int_{-\infty}^{+\infty} dr e^{4A} \left\{ 12 (A' \mp g W)^2 - (\sigma' \pm 3g \partial_\sigma W)^2 - g_{xy} (\phi^{x'} \pm 3g \partial^x W) (\phi^{y'} \pm 3g \partial^y W) \right. \\
& + 36W^2 \partial_\sigma Q_{\underline{ab}} \partial_\sigma Q^{\underline{ab}} + 18|\tilde{X}| (\partial_\sigma X_{\underline{a}})^2 - \frac{1}{2} \Sigma^4 (\xi^{aa} X_{\underline{a}})^2 - 4\Sigma^{-2} \text{Tr} [f^{a\bar{a}b} L^r_{\underline{ab}} (\delta^{rs} + Q^r Q^s)]^2 \Big\} \\
& + [e^{4A} (\pm 3g W - 4A')]_{-\infty}^{+\infty}.
\end{aligned} \tag{3.58}$$

Since it is an algebraic sum of squares of the supersymmetry conditions summarized in section 3.2, clearly it is extremized by them. It follows that the Einstein and scalar equations of motion of five-dimensional $\mathcal{N} = 4$ supergravity are solved. In order to have a full solution, it remains to consider the equations of motion for the vector and two-form fields in the theory. The equations for the two-forms are trivially satisfied in the domain wall background. As for the equations for the vector fields, the only non-trivial contribution comes from the scalar kinetic term $g_{xy} D_\mu \phi^x D^\mu \phi^y$, where the gauge covariant derivative is $D_\mu \phi^x = \partial_\mu \phi^x + \Sigma^{-2} K^x_\mu A^0 + \Sigma K^x_M A^M$. Here, A^0 and A^M are the $\mathcal{N} = 4$ vector fields, while the $\Sigma^{-2} K$ and ΣK_M are the Killing vectors on the scalar manifold which generate the isometries being gauged (see appendix B for their explicit expression). The vector equations then read

$$\phi^{x'} K_x = 0, \quad \phi^{x'} K^M_x = 0, \tag{3.59}$$

which, using the flow equations, coincide with the requirement of gauge invariance of the superpotential W . Hence we conclude that for a gauge-invariant superpotential also the vector field equations are satisfied and our conditions yield a solution to $\mathcal{N} = 4$ supergravity.

4 Deforming SCFT's on D3-branes at Calabi-Yau cones

We now apply the results of our general analysis, in order to study supersymmetric domain wall solutions, to two specific $\mathcal{N} = 4$ supergravity models, which arise as consistent truncations of type IIB supergravity and are relevant for the gauge/gravity correspondence. The first, to be studied in this section, can be defined as a dimensional reduction on any five-dimensional Sasaki–Einstein manifold, while the second, which is an extension of the former, is based on the $T^{1,1}$ coset space, and will be discussed in the next section.

In [18] (see also [19, 28, 29, 30, 31]), a universal consistent truncation of type IIB supergravity on five-dimensional manifolds admitting a Sasaki–Einstein (SE) structure was constructed, leading to gauged $\mathcal{N} = 4$ supergravity in five dimensions. This model admits the standard AdS_5 solution of type IIB on a Sasaki–Einstein manifold, and is thus suitable

to describe universal deformations of the superconformal field theory on D3-branes probing the tip singularity of the Calabi–Yau cone over the Sasaki–Einstein space. It comprises two $\mathcal{N} = 4$ vector multiplets and hence eleven scalar fields, dubbed $U, V, \phi, C_0, a, b^J, c^J$ (real) and b, c (complex) in [18].⁶ Here, C_0 and ϕ are the axion and dilaton of type IIB supergravity, while the other scalars enter in the ten-dimensional metric as

$$ds_{10}^2 = e^{-\frac{2}{3}(4U+V)+2A} ds^2(\mathbb{R}^{1,3}) + e^{-\frac{2}{3}(4U+V)} dr^2 + e^{2U} ds^2(\mathcal{B}_{\text{KE}}) + e^{2V} \eta^2, \quad (4.1)$$

in the type IIB two-form potentials as

$$B = b^J J + \text{Re}(b \Omega), \quad C_2 = c^J J + \text{Re}(c \Omega), \quad (4.2)$$

and in the five-form field strength as

$$F_5 = (1 + *_{10}) \left\{ [3 \text{Im}(b \bar{c}) + k] J \wedge J \wedge \eta + \frac{1}{2} [2da + b^J dc^J - c^J db^J + \text{Re}(b d\bar{c} - c d\bar{b})] \wedge J \wedge J \right\}. \quad (4.3)$$

\mathcal{B}_{KE} is the four-dimensional Kähler–Einstein base of the Sasaki–Einstein manifold, J is its Kähler form, η is the contact one-form dual to the Reeb vector, satisfying $d\eta = 2J$, and Ω is the complex two-form which satisfies $d\Omega = 3i(\eta \wedge \Omega)$. The constant k parameterizes the five-form flux. Note that in the metric we already assumed a five-dimensional domain wall ansatz.

The scalar in the $\mathcal{N} = 4$ gravity multiplet is identified with $\sigma = -\frac{2}{\sqrt{3}}(U + V)$, while the others parameterize the $\frac{\text{SO}(5,2)}{\text{SO}(5) \times \text{SO}(2)}$ coset space, see [18] for details. In the parameterization chosen there, the embedding tensor components are

$$\begin{aligned} f_{125} = f_{256} = f_{567} = -f_{157} = -2, \\ \xi_{34} = -3\sqrt{2}, \quad \xi_{12} = \xi_{17} = -\xi_{26} = \xi_{67} = -\sqrt{2}k, \end{aligned} \quad (4.4)$$

and their permutations. For the non-vanishing components of the gravitino shift matrix $P_{ab} = -P_{ba}$ defined in (2.17) we find

$$\begin{aligned} P_{12} &= \frac{1}{6} e^{-\frac{16}{3}U - \frac{4}{3}V} [3 \text{Im}(b \bar{c}) + k], & P_{34} &= \frac{1}{6} e^{-\frac{10}{3}U - \frac{4}{3}V} (3e^{2U} + 2e^{2V}) \\ P_{13} &= \frac{1}{2} e^{-\frac{10}{3}U - \frac{4}{3}V + \frac{1}{2}\phi} \text{Im}(c - C_0 b), & P_{14} &= -\frac{1}{2} e^{-\frac{10}{3}U - \frac{4}{3}V + \frac{1}{2}\phi} \text{Re}(c - C_0 b), \\ P_{23} &= \frac{1}{2} e^{-\frac{10}{3}U - \frac{4}{3}V - \frac{1}{2}\phi} \text{Im} b, & P_{24} &= -\frac{1}{2} e^{-\frac{10}{3}U - \frac{4}{3}V - \frac{1}{2}\phi} \text{Re} b. \end{aligned} \quad (4.5)$$

This yields

$$X_a = \delta_a^5, \quad (4.6)$$

⁶ b, c were actually denoted b^Ω, c^Ω in [18].

so we can choose the $\text{su}(2)_\pm$ generators as in footnote 3.

We can then evaluate the constraints for a supersymmetric domain wall found in the general analysis. The first condition in (3.29) is trivial, and it turns out that also constraints (3.31), (3.32) are automatically satisfied. The first in (3.30) instead is non-trivial (because of the piece containing the $\text{SO}(5)$ connection) and tells that

$$b^J = \text{const}, \quad c^J = \text{const}. \quad (4.7)$$

The second in (3.29), namely $\partial_\sigma Q_\pm^r = 0$, gives

$$c = (C_0 \pm i e^{-\phi}) b, \quad (4.8)$$

where the two sign choices are correlated. We also compute W_\pm and find the associated flow equations. One can check that these are compatible with (4.7), (4.8), and that the equations for the remaining scalars $\{U, V, \phi, C_0, a, b\}$ consistently follow from the reduced superpotential and the reduced metric that are found by plugging (4.7), (4.8) in the original W_\pm and in the scalar kinetic terms. It is therefore enough to present this reduced superpotential, which reads

$$W_\pm = e^{-\frac{4}{3}U - \frac{4}{3}V} + \frac{2}{3} e^{-\frac{10}{3}U + \frac{2}{3}V} + e^{-\frac{16}{3}U - \frac{4}{3}V} \left(e^{-\phi} |b|^2 \mp \frac{1}{3}k \right), \quad (4.9)$$

together with the reduced metric, which is

$$\begin{aligned} g_{xy} d\phi^x d\phi^y|_{\text{red}} &= 8(dU)^2 + \left(\frac{1}{2} + e^{-4U - \phi} |b|^2 \right) [e^{2\phi} (dC_0)^2 + (d\phi)^2] \\ &\quad + 2e^{-4U} \left[e^{-\phi} |db|^2 \pm \text{Im}(b d\bar{b}) dC_0 - \frac{1}{2} e^{-\phi} d(|b|^2) d\phi \right] \\ &\quad + 2e^{-8U} \left[da + \frac{1}{2} |b|^2 dC_0 \pm e^{-\phi} \text{Im}(b d\bar{b}) \right]^2. \end{aligned} \quad (4.10)$$

Choosing the upper sign in (3.27), (3.28), the associated flow equations read:

$$\begin{aligned} U' &= e^{-\frac{10}{3}U + \frac{2}{3}V} + \frac{1}{2} e^{-\frac{16}{3}U - \frac{4}{3}V} \left(3e^{-\phi} |b|^2 \mp k \right), \\ V' &= 3e^{-\frac{4}{3}U - \frac{4}{3}V} - 2e^{-\frac{10}{3}U + \frac{2}{3}V} + \frac{1}{2} e^{-\frac{16}{3}U - \frac{4}{3}V} \left(3e^{-\phi} |b|^2 \mp k \right), \\ b' &= -3e^{-\frac{4}{3}U - \frac{4}{3}V} b, \\ \phi' &= C'_0 = a' = 0. \end{aligned} \quad (4.11)$$

This system can be solved by introducing the new radial coordinate

$$d\rho = e^{-\frac{4}{3}(U+V)} dr, \quad (4.12)$$

so that the equations for b and $(U - V)$ decouple. The general solution is

$$\begin{aligned} b &= b_0 e^{-3\rho}, \\ e^{2(U-V)} &= 1 + \alpha e^{-6\rho}, \\ e^{2(U+V)} &= \frac{\beta e^{4\rho}}{(1 + \alpha e^{-6\rho})^{\frac{1}{3}}} - \frac{|b_0|^2}{e^{\phi}\alpha} + \left(\frac{|b_0|^2}{e^{\phi}\alpha} \pm \frac{k}{2} \right) {}_2F_1 \left[1, 1, \frac{5}{3}; -\alpha e^{-6\rho} \right], \end{aligned} \quad (4.13)$$

where b_0 (complex) and α, β (real) are integration constants. The hypergeometric function appearing in the last line is

$${}_2F_1 \left[1, 1, \frac{5}{3}; z \right] = \frac{Z}{3z} \left[2\sqrt{3} \left(\frac{\pi}{2} + \arctan \frac{Z-2}{\sqrt{3}Z} \right) + \log \frac{1-Z+Z^2}{(1+Z)^2} \right], \quad Z = \left(\frac{z}{1-z} \right)^{\frac{1}{3}}. \quad (4.14)$$

In the variable ρ , the equation for the domain wall warp factor A becomes

$$A' = 1 + \frac{2}{3} U' \quad \Rightarrow \quad A = \rho + \frac{2}{3} U, \quad (4.15)$$

where in its solution we are neglecting an integration constant, as it can always be reabsorbed in a rescaling of the four-dimensional Minkowski coordinates.

When $\beta = 0$, as $\rho \rightarrow +\infty$ the scalars take a finite value which extremizes the five-dimensional scalar potential, and the metric is asymptotically AdS_5 (with radius $L^4 = |k|/2$). In this case, noting that

$$\lim_{z \rightarrow 0} {}_2F_1 \left[1, 1, \frac{5}{3}; z \right] = 1, \quad (4.16)$$

we see that positivity of $e^{2(U+V)}$ as $\rho \rightarrow +\infty$ imposes $k > 0$ for the upper sign choice and $k < 0$ for the lower choice. Moreover, positivity of $e^{2(U-V)}$ requires $e^{6\rho} > -\alpha$.

We checked that all the five-dimensional equations of motion given in [18] are satisfied, as expected from the general analysis in section 3. Due to the consistency of the truncation, the five-dimensional solution lifts to a solution of type IIB supergravity. It is interesting to discuss how it looks like in ten dimensions. From the reduction ansatz given at the beginning of this section, we find that the ten-dimensional metric takes the following D3-brane form

$$ds^2 = h^{-\frac{1}{2}}(\rho) ds^2(\mathbb{R}^{1,3}) + h^{\frac{1}{2}}(\rho) ds^2(M_6) \quad (4.17)$$

with warp factor

$$h(\rho) = \beta + e^{-4\rho} (1 + \alpha e^{-6\rho})^{\frac{1}{3}} \left[-\frac{|b_0|^2}{e^{\phi}\alpha} + \left(\frac{|b_0|^2}{e^{\phi}\alpha} \pm \frac{k}{2} \right) {}_2F_1 \left[1, 1, \frac{5}{3}; -\alpha e^{-6\rho} \right] \right] \quad (4.18)$$

and six-dimensional transverse metric

$$ds^2(M_6) = \frac{e^{2\rho}}{(1 + \alpha e^{-6\rho})^{\frac{2}{3}}} \left[(d\rho)^2 + \eta^2 + (1 + \alpha e^{-6\rho}) ds^2(\mathcal{B}_{\text{KE}}) \right]. \quad (4.19)$$

For $b_0 = 0$, this solution had already appeared in [20, 21] (see also [32, app. B]).⁷ The metric (4.19), first found in [20] in the case of the conifold, is Calabi–Yau and describes a Calabi–Yau cone deformed by blowing-up the four-cycle \mathcal{B}_{KE} [21]. The deformation is controlled by α , with the cone being retrieved as $\alpha = 0$. In particular, if both α and β vanish, the standard $\text{AdS}_5 \times \text{SE}_5$ solution of type IIB supergravity is obtained. When $b_0 \neq 0$, our solution has an extra feature: the complexified three-form of type IIB supergravity is non-trivial and reads

$$G_3 := i e^{\frac{\phi}{2}} [i e^{-\phi} dB - (dC_2 - C_0 dB)] = 3 b_0 e^{-\frac{\phi}{2} - 3\rho} \Omega \wedge (d\rho - i\eta) \quad (4.20)$$

for the lower sign choice in (4.18), and its complex conjugate for the upper choice. This is closed, imaginary (anti-)self-dual and primitive with respect to the Calabi–Yau structure. So we conclude that the present $\mathcal{N} = 1$ solution falls in the well-known class of supersymmetric warped Calabi–Yau backgrounds of type IIB supergravity with non-trivial three-form flux and constant axio-dilaton, first discussed in [33].

We also observe that for $b_0 \neq 0$ one can take a limit which, when the Sasaki–Einstein manifold is S^5 , corresponds to the $\text{SU}(3)$ -invariant sector of the GPPZ flow [22] (see [34] for its lift to type IIB supergravity). This is most easily seen by choosing the five-form flux so that $k = 2$. Then, with $\beta = 0$ and relating the other integration constants as $|b_0|^2 = -\alpha e^\phi$ (so we need $\alpha < 0$), the third line of (4.13) simplifies, and the scalars with a non-trivial profile can be parameterized by the single function

$$\lambda(\rho) = \frac{1}{2} \log \left[\frac{1 + (-\alpha)^{1/2} e^{-3\rho}}{1 - (-\alpha)^{1/2} e^{-3\rho}} \right] \quad (4.21)$$

as

$$\begin{aligned} -U &= V = \frac{1}{2} \log (\cosh \lambda), \\ \left| \frac{c}{C_0 + i e^{-\phi}} \right| &= |b| = e^{\phi/2} \tanh \lambda. \end{aligned} \quad (4.22)$$

The superpotential reduces to $W = \cosh^2 \lambda$, and the flow equations to $\lambda' = -3 \partial_\lambda W$. This precisely describes the $\text{SU}(3)$ -invariant sector of the GPPZ flow. We can thus conclude that tuning $|b_0|^2$ from 0 to $-\alpha e^\phi$, our solution interpolates between the solution of [20, 21] and the one of [22].

Finally, from the asymptotic behaviour of the scalar fields we can identify which operators in the dual field theory are driving the RG flow. Denoting by \mathcal{W} the gauge superfield of the $\mathcal{N} = 1$ SCFT dual to type IIB on $\text{AdS}_5 \times \text{SE}_5$, we find (see for instance [18]) that we

⁷The change of variable leading to the expressions given there is $(1 + \alpha e^{-6\rho})^{-1} = 1 - \frac{b^6}{r^6}$, with $b^2 = \alpha^{1/3}$.

are switching on a vev for the first components of $\text{Tr}(\mathcal{W}^2)$ and of $\text{Tr}(\mathcal{W}^2\overline{\mathcal{W}}^2)$. The former is a relevant operator with conformal dimension $\Delta = 3$, which drives the GPPZ flow and is dual to (the fluctuation of) $b - ic$. The latter has $\Delta = 6$, so it is irrelevant, and is dual to $U - V$. Setting $\beta = 0$ we have eliminated a possible further deformation, corresponding to an irrelevant source D-term $\int d^2\theta d^2\bar{\theta} \text{Tr}(\mathcal{W}^2\overline{\mathcal{W}}^2)$, which has $\Delta = 8$ and is dual to $4U + V$ (being an irrelevant source in the Lagrangian, this cannot be switched on as long as the CFT is regarded as the UV theory, and correspondingly the asymptotic AdS factor on the gravity side is preserved).

5 New superpotentials for conifold solutions

In the construction of string theory solutions dual to non-trivial RG flows, a seminal role was played by the conifold [35]. This is a Calabi–Yau three-fold constructed as the cone on the coset space $T^{1,1} = [\text{SU}(2) \times \text{SU}(2)]/\text{U}(1)$, endowed with its Sasaki–Einstein metric. The superconformal field theory dual to type IIB on $\text{AdS}_5 \times T^{1,1}$, which describes the low energy dynamics of D3-branes probing the tip singularity of the conifold, was identified in [36]. The addition of fractional D3-branes breaks conformality and induces RG flows with non-trivial properties such as duality cascades, confinement and chiral symmetry breaking [37, 3, 4, 6, 7]. Similar features are obtained by wrapping D5-branes on the two-cycle of $T^{1,1} \sim S^2 \times S^3$ [5]. In the corresponding type IIB supergravity solutions, the cone geometry gets deformed, in a way that also affects the metric on $T^{1,1}$. All the conifold solutions [3, 4, 5, 6, 7] preserve four-dimensional Poincaré invariance and can be seen from a five-dimensional perspective as domain walls of the type studied in this paper. Not all the relevant $T^{1,1}$ deformations are captured by the universal Sasaki–Einstein truncation considered in the previous section. However, in [13] (see also [15, 38]), a larger $\mathcal{N} = 4$ consistent truncation containing all known conifold solutions was constructed by retaining all (and only) the Kaluza–Klein modes of type IIB supergravity on $T^{1,1}$ which are invariant under the action of $\text{SU}(2) \times \text{SU}(2)$. This adds an $\mathcal{N} = 4$ vector multiplet to the universal Sasaki–Einstein truncation, and introduces further non-trivial gaugings, which in type IIB correspond to background three-form fluxes threading the three-cycle of $T^{1,1}$.

In the following, we scan for supersymmetric, $\text{SU}(2) \times \text{SU}(2)$ invariant conifold solutions by applying the general results of section 3 to the $\mathcal{N} = 4$ supergravity model of [13]. We perform an exhaustive search within a subset of the $\mathcal{N} = 4$ scalars, originally introduced by Papadopoulos and Tseytlin (PT) in [23], which is sufficient for describing all known conifold solutions. Note that the results of [13] are anyway essential in order to identify the $\mathcal{N} = 4$ structure of the PT ansatz. Besides recovering the known superpotentials and flow

equations for the solutions in [3, 4, 5], we find the superpotential for the solution describing the baryonic branch of the Klebanov–Strassler theory [7], which was unknown before and generalizes the previous ones. In addition, we obtain the superpotential for the solution describing supersymmetric D3- and wrapped D5-branes on the resolved conifold.

5.1 Superpotential for the baryonic branch of Klebanov-Strassler

The PT ansatz [23] (see also [39] for its formulation in five dimensions) contains nine scalars, denoted by p, x, g, a (from the internal metric), b (from the RR two-form potential), h_1, h_2, χ (from the NSNS two-form potential) and ϕ (the dilaton). Moreover, it contains two parameters P and Q , corresponding to RR three-form and five-form fluxes, respectively. As in the previous section, we proceed by constructing the $\mathcal{N} = 4$ fermionic shifts using [13], then we evaluate the algebraic constraints, and eventually write down the superpotential and the flow equations, checking compatibility with the constraints. Since in the present case the intermediate steps are considerably more involved than in the previous section, we will just give the results.⁸

For the PT system, constraint (3.31) is automatically satisfied, while the first condition in (3.29) demands that either

$$a^2 + e^{2g} = 1 \tag{5.1}$$

or

$$(1 + a^2 + e^{2g})h_2P + a(2h_1P + Q) = 0. \tag{5.2}$$

The first possibility eventually leads to just the warped deformed conifold solution of [4], so we will not discuss it further. Let us instead study condition (5.2), which has more interesting consequences. The subcase $a = 0$ is special and will be discussed in the next subsection. Here we assume $a \neq 0$ and $P \neq 0$, and use (5.2) to fix h_1 in terms of the other fields. Then the second constraint in (3.29) provides an expression that can be used to determine e.g. h_2 .

⁸For the ease of comparison with the literature, we give our results adopting the normalizations of [23]. It follows that the flow equations read

$$A' = -\frac{1}{3}W, \quad \varphi^{a'} = \frac{1}{2}G^{ab}\frac{\partial W}{\partial \varphi^b},$$

where φ^a are the scalars in the PT ansatz and G_{ab} is the kinetic matrix in [23]. Once our algebraic constraints are taken into account, the relation between the scalar potential V of [23] and the superpotential is

$$V = \frac{1}{8}G^{ab}\frac{\partial W}{\partial \varphi^a}\frac{\partial W}{\partial \varphi^b} - \frac{1}{3}W^2.$$

The scalar in the supergravity multiplet is identified as $\sigma = 2\sqrt{3}p$. For the map between the other $\mathcal{N} = 4$ fields and the PT ones, see appendix B of [13].

We find:

$$h_1 = -C h_2 - \frac{Q}{2P}, \quad (h_2)^2 = \frac{e^{2\phi} P^2 (bC - 1)^2 - e^{2x-2g+\phi} (aC - 1)^2}{S^2}, \quad (5.3)$$

where we introduced the following functions of a and g

$$C \equiv \frac{1 + a^2 + e^{2g}}{2a}, \quad (5.4)$$

$$S \equiv \frac{\sqrt{a^4 + 2a^2(-1 + e^{2g}) + (1 + e^{2g})^2}}{2a}, \quad (5.5)$$

which satisfy $C^2 - S^2 = 1$. One can see that in this way all constraints in (3.2) are satisfied. Plugging (5.3) back into W computed as in (3.15), we obtain the following superpotential:

$$W = e^{-2p-2x-g} a S + e^{4p} S^{-1} [C + e^{-2x+\phi} P^2 (b - C) (bC - 1)]. \quad (5.6)$$

One can check that via the relation in footnote 8 this reproduces the PT scalar potential with h_1 and h_2 integrated out as dictated by (5.3). The flow equations following from (5.6) read

$$\begin{aligned} C' &= S \\ b' &= \frac{1 - bC}{S} \\ a' &= \frac{1 - aC}{S} e^{g-6p-2x} + \frac{a(a-b)S}{bC-1} \\ x' &= a S e^{-g-6p-2x} + \frac{b-C}{2(bC-1)S} [e^{-2g}(aC-1)^2 + 2e^{-2x-\phi}(h_2)^2 S^2] \\ 6p' &= a S e^{-6p-2x-g} - \frac{2(b-C)}{(bC-1)S} [e^{-2g}(aC-1)^2 + e^{-2x-\phi}(h_2)^2 S^2] - \frac{2C}{S} \\ \phi' &= \frac{(C-b)(aC-1)^2}{(bC-1)S} e^{-2g} \\ \chi' &= \frac{2a(b-C)(aC-1)}{bC-1} e^{-2g} h_2 S. \end{aligned}$$

Here, the derivative is with respect to the rescaled radial coordinate $dt = e^{4p} du$, where u is the PT coordinate. Moreover, we traded g for C using (5.4), and h_2 is to be replaced by (5.3). The kinetic matrix used in the formula for the flow is the one obtained by plugging (5.3) in the kinetic terms of [23]. The equations above are precisely those for the solution describing the baryonic branch of the Klebanov–Strassler theory, which were obtained in [7] by other

means, working in ten dimensions.⁹ Hence we have found a superpotential for that solution, which was unknown before. This provides a monotonic c -function along the flow.

Starting from our expression (5.6), the known superpotentials for the Maldacena–Nuñez [5] and the Klebanov–Strassler [4] solutions can be recovered as limiting cases once one restricts to the respective systems of variables. The Maldacena–Nuñez variables are obtained from the PT ones by setting [39]:

$$\chi = h_1 = h_2 = Q = 0, \quad b = a, \quad \phi = -6p - g - 2 \log P, \quad x = \frac{1}{2}g - 3p, \quad (5.7)$$

so (5.3) are consistently satisfied and W becomes

$$W_{\text{MN}} = 2 e^{4p-2g} a S = e^{4p} \sqrt{(a^2 - 1)^2 e^{-4g} + 2(a^2 + 1) e^{-2g} + 1}, \quad (5.8)$$

which is the superpotential of the Maldacena–Nuñez solution, found in [23]. In order to recover the Klebanov–Strassler system, we parameterize a and g as

$$a = \tanh y, \quad e^{-g} = \cosh y, \quad (5.9)$$

so that $C = \coth y$, and (5.3) becomes

$$h_1 = -h_2 \coth y - \frac{Q}{2P}, \quad (h_2)^2 = e^{2\phi} P^2 (b \cosh y - \sinh y)^2, \quad (5.10)$$

while W in (5.6) reads

$$W_{\text{KS}}|_{h_1, h_2} = e^{-2p-2x} + e^{4p} \cosh y + e^{4p-2x+\phi} P^2 (b \cosh y - \sinh y) (b - \coth y). \quad (5.11)$$

This is the known superpotential for the Klebanov–Strassler solution, namely [6]

$$W_{\text{KS}} = e^{-2p-2x} + e^{4p} \cosh y - \frac{1}{2} e^{4p-2x} (Q + 2P h_1 + 2P b h_2), \quad (5.12)$$

with h_1, h_2 replaced by (5.10).

Another limiting case that can be considered is the one in which both the NSNS three-form and the RR five-form vanish, yielding solutions describing just wrapped D5-branes. This setup was studied in [40], and a superpotential for it was found in [41, sect. 3]. In the PT ansatz it corresponds to the truncation $h_2 = 0$, $h_1 = -Q/(2P)$ and $\chi = \text{const}$. Using the second in (5.3), one can check that then our superpotential reduces to the one in [41].¹⁰

⁹When comparing with [7, app. B], one has to recall that they use the string frame, while as [23] we work in the Einstein frame. Comparing the respective ten-dimensional metric ansätze, one finds $x_{\text{there}} = x_{\text{here}} + \phi/2$, $p_{\text{there}} = p_{\text{here}} - \phi/6$ and $A_{\text{there}} = (A + p - x/2 + \phi/4)_{\text{here}}$. Moreover, in [7] the three-form flux parameter was fixed to $P = -1/2$, while we leave it arbitrary.

¹⁰The change of variables between [41] and PT is

$$\begin{aligned} P_{\text{HNP}} &= 4 a S e^{x-g-\frac{\phi}{2}}, & Q_{\text{HNP}} &= 4 (a C - 1) e^{x-g-\frac{\phi}{2}}, \\ Y &= e^{-6p-x-\frac{\phi}{2}}, & \sinh \tau &= S^{-1}, & N_c &= 4P, & N_f &= 0, \end{aligned}$$

and their constraint $\omega = 0$ is the same as the second in our (5.3), with $h_2 = 0$.

5.2 Supersymmetric D3 and D5-branes on resolved conifold

In the following we study the case $a = 0$, which was left aside in the previous section and allows for a different solution of the constraints leading to a different superpotential. Geometrically, $a = 0$ means that there are no mixed terms in the metric on the $\mathbb{CP}^1 \times \mathbb{CP}^1$ base of the $U(1)$ -fibration describing $T^{1,1}$. With $a = 0$, constraint (5.2) implies $h_2 = 0$ (we are interested in solutions with $P \neq 0$). Then the remaining algebraic conditions of section 3.2 also require $b = 0$. We are thus led to consider an $a = b = h_2 = 0$ truncation of the PT ansatz, which corresponds to the ansatz first considered by Pando-Zayas and Tseytlin in [6] in order to study solutions based on the resolved conifold geometry.¹¹ The type IIB supergravity solution found there, describing regular D3 and wrapped D5-branes on the resolved conifold, is known to be non-supersymmetric [42] (see also [43]), although it can be described in terms of a simple superpotential. In the following, applying our general method we identify a supersymmetric solution describing D3 and wrapped D5-branes on the resolved conifold, having the same active fields as in [6], plus the dilaton. For the superpotential we find

$$W = e^{-2p-2x} \cosh g + e^{4p} + \frac{1}{2} e^{4p-2x} \sqrt{4 e^{2x+\phi} P^2 \sinh^2 g + (Q + 2Ph_1)^2}. \quad (5.13)$$

This is similar to the one given in [6], the latter being recovered by erasing the first term in the square root. Both superpotentials define BPS domain walls in that both reproduce the same truncated PT potential $V|_{a=b=h_2=0}$ through the relation in footnote 8. However, only the one in (5.13) generates a supersymmetric flow. Note in particular that the new term introduces a dependence on the dilaton; as a consequence, ϕ will flow in our solution.

We present the system of first-order equations for the fields $\{x, g, p, \phi, h_1, \chi\}$ following from (5.13) in the radial coordinate ρ , related to the PT one by $d\rho = e^{4p} du$. We find:

$$\begin{aligned} x' &= -e^{-6p-2x} \cosh g - \frac{1}{2} e^{-2x} T^{-1} [(Q + 2Ph_1)^2 + 2 e^{2x+\phi} P^2 \sinh^2 g] \\ g' &= e^{-6p-2x} \sinh g + e^\phi P^2 T^{-1} \sinh(2g) \\ \phi' &= 2 e^\phi P^2 T^{-1} \sinh^2 g \\ p' &= \frac{1}{3} - \frac{1}{6} e^{-6p-2x} \cosh g + \frac{1}{6} e^{-2x} T \\ h_1' &= e^\phi P T^{-1} \cosh(2g) (Q + 2Ph_1) \\ \chi' &= e^\phi P T^{-1} \sinh(2g) (Q + 2Ph_1), \end{aligned} \quad (5.14)$$

¹¹The change of variables leading to the notation in [6] is $h_1 = \frac{1}{2}(f_1 - f_2)$, $\chi = \frac{1}{2}(f_1 + f_2)$, $g = y$.

where we introduced the quantity

$$T = \sqrt{(Q + 2Ph_1)^2 + 4e^{2x+\phi}P^2 \sinh^2 g}. \quad (5.15)$$

Setting the three-form flux $P = 0$, one recovers the equations describing regular D3-branes on the resolved conifold [6]. Linearizing for small P , which physically means taking a small ratio between the three-form and five-form Page charges, we recover precisely equations (5.15)–(5.18) of [6]. However, at higher orders the solution deviates from the one in [6].

The system can be partially solved by first noting that

$$(e^{-\phi}T)' = 2P^2 \quad \Rightarrow \quad e^{-\phi}T = 2P^2\rho, \quad (5.16)$$

where we are absorbing the integration constant in a shift of ρ . This eliminates T from the equations. We also introduce the new convenient variables $w = x - \phi/2$ and $z = x + 3p$. The solution for ϕ is

$$\phi = \frac{z}{2} + \frac{3}{4} \log \left| \frac{\sinh g}{2P^2\rho} \right| - \frac{\rho}{2} + \gamma, \quad (5.17)$$

with γ an integration constant. Moreover we find that w is determined by

$$e^{2w} \sinh^2 g = (1 - \kappa^2 e^{2\phi}) P^2 \rho^2, \quad (5.18)$$

where κ is a constant such that $\kappa^2 \leq e^{-2\phi}$, and that h_1 is fixed by

$$Q + 2Ph_1 = 2\kappa P^2 e^{2\phi} \rho. \quad (5.19)$$

The warp factor reads

$$A = \frac{1}{3}(\rho - w). \quad (5.20)$$

We are thus left with three equations:

$$\begin{aligned} g' &= e^{-2z} \sinh g + \frac{\sinh(2g)}{2\rho}, \\ z' &= 1 - \frac{3}{2} e^{-2z} \cosh g + \frac{\sinh^2 g}{2\rho}, \\ \chi' &= \kappa P e^{2\phi} \sinh(2g), \end{aligned} \quad (5.21)$$

where just the first two need to be solved since knowing the derivative of χ in the PT ansatz is enough to determine the ten-dimensional background.

This reduced system is not new. From (5.19) we observe that $\kappa = 0$ implies $Q + 2Ph_1 = 0$, which in the present ansatz means that the type IIB five-form F_5 vanishes, i.e. there is no

D3-brane charge and we are left just with a D5-brane setup. In this case, the equations have already been studied in [40]. The solution for $\kappa \neq 0$ is instead a particular case of the one discussed in [44] when one takes $a = b = 0$. In [44], a U-duality transformation was proposed mapping a solution with just D5 charge into a solution with both D3 and D5-charge. If we denote by w_0 the solution (5.18) in the absence of D3-brane charge and thus with $\kappa = 0$, then the general solution reads

$$e^{2w} = (1 - \kappa^2 e^{2\phi}) e^{2w_0}. \quad (5.22)$$

This is precisely the form of the transformation described in [44, 45], where the dilaton and the metric functions g, z are unaltered. Moreover, our D3-brane charge (5.19) and the χ equation also agree with their formulae. We conclude that any solution to the system (5.14) can be obtained from a solution with just D5-brane charge via the transformation of [44].

Note that the equations can easily be solved analytically if the special solution $g = 0$ is chosen, meaning that the two \mathbb{CP}^1 's in $T^{1,1}$ have the same size. This leads to the superpotential and the equations given by Klebanov and Tseytlin in [3]. In this case χ and the dilaton ϕ are constant and the general solution for z is $e^{2z} = \frac{3}{2} (1 + z_0 e^{2\rho})$. Selecting the integration constant as $z_0 = 0$ one recovers the solution of [3] describing regular and fractional D3-branes on the singular conifold, whereas for $z_0 \neq 0$ the solution was given in [20]. Here we explicitly exhibit its supersymmetric nature. The transverse geometry is the same Calabi–Yau as in (4.19), while the warp factor h is more complicated than the one in section 4, though still expressed in terms of hypergeometric functions [20].

It is interesting to look at the form taken by the constraints on the spinorial susy parameter discussed in section 3.2. These can be lifted to IIB supergravity using the map between ten-dimensional and five-dimensional fermionic variations derived in [38]. In their spinor conventions, we find that the type IIB supersymmetry parameters are constrained as¹²

$$\begin{aligned} \epsilon - \frac{1}{8} \Gamma^{0123} \Gamma^{AB} J_{AB} \epsilon &= 0, \\ \epsilon - i \Gamma^{0123} (\cos \beta \epsilon + \frac{1}{8} \sin \beta \Gamma^{AB} J_{AB} \epsilon^c) &= 0, \end{aligned} \quad (5.23)$$

where $\cos \beta = \kappa e^\phi$. The first projector is compatible with D5-branes filling the four flat spacetime directions and wrapping the submanifold in $T^{1,1}$ identified by the Kähler form

¹²In five-dimensional language we have

$$\begin{aligned} \varepsilon - \gamma^5 \sigma_3 \varepsilon &= 0, \\ \varepsilon - \gamma^5 (\cos \beta \varepsilon - \sin \beta \sigma_1 \varepsilon^c) &= 0, \end{aligned}$$

where $\varepsilon = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}$ is an $\text{Sp}(2)$ doublet of Dirac spinors as in [38]. In this notation, the type IIB Weyl supersymmetry parameter is given by $\epsilon_{\text{IIB}} = \varepsilon_1 \otimes \eta + \varepsilon_2 \otimes \eta^c$, with η being a Killing spinor on the $T^{1,1}$ and η^c its charge conjugate.

J on $\mathbb{CP}^1 \times \mathbb{CP}^1$. The second projector describes D3-branes filling the four flat spacetime dimensions and polarized into the D5-branes through the Myers effect [46]; it is of the type considered in [47].

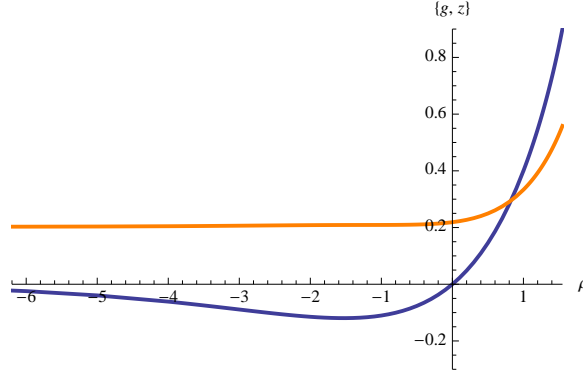


Figure 1: Metric functions $g(\rho)$ (dark blue) and $z(\rho)$ (light orange) for the initial values $g(1) = 2/5$, $z(1) = 1/3$. For the fluxes, we chose $P = Q = 1$. The functions asymptote to $g = 0$ and $z = \frac{1}{2} \log \frac{3}{2}$ when $\rho \rightarrow -\infty$, which are the values of the singular conifold solution in [3]. The singularity appears at $\rho \approx 1.55$.

We conclude this section by presenting in figure 1 a numeric solution to the g and z equations in (5.21). We start the integration at an arbitrary point ρ_0 , using a power series solution around it as initial condition. The system is then integrated both to the left and right of the starting point. We have not explored thoroughly the space of parameters, but an IR singularity appears in all the solutions we generated. This is consistent with the vanishing of a and b modes that are crucial for avoiding the singularity. In the UV, corresponding to $\rho \rightarrow -\infty$, the behaviour of the conifold solution in [3] is recovered, and at first-order the metric functions g and z go as

$$g \sim \rho^P e^{\frac{2}{3}\rho}, \quad z \sim \frac{1}{2} \log \frac{3}{2} + e^{2\rho}. \quad (5.24)$$

The transverse space is not Ricci-flat when evaluated in our numeric solutions. Furthermore, the scalar curvature contains an IR singularity.

6 Conclusions

Domain walls in supergravity theories are holographically dual to field theory RG flows and are thus interesting solutions to explore. In five-dimensional, $\mathcal{N} = 2$ supergravity the problem of constructing BPS domain walls was addressed some time ago [10, 11] and used for instance to recover the superpotentials of some conifold solutions [12]. However, the $\mathcal{N} = 2$

approach is in this case limited and one needs to go to $\mathcal{N} = 4$ in order to supersymmetrize the complete PT ansatz and all the solutions it encompasses.

In this paper, we have presented a compact set of necessary and sufficient conditions for supersymmetric domain wall solutions to $\mathcal{N} = 4$ gauged supergravity in five dimensions. As summarized in section 3.2, these take the form of first-order flow equations for the warp factor and the scalar fields, generated by a superpotential and completed by a set of algebraic constraints involving the scalars and the embedding tensor specifying the gauging of the $\mathcal{N} = 4$ model. We found that supersymmetric domain walls are generically 1/4 BPS, namely they preserve four supercharges.

We subsequently applied our general conditions to two particular $\mathcal{N} = 4$ consistent truncations of type IIB supergravity (so the solutions are automatically embedded into string theory). The first is based on a reduction on an internal manifold with Sasaki–Einstein structure [18, 19]. We gave the explicit solution for its most general supersymmetric domain wall, thus describing a universal RG flow deforming superconformal theories on D3-branes at Calabi–Yau cones. Its ten-dimensional uplift contains a primitive imaginary-self-dual three-form, and tuning the associated parameter we interpolate between the flows in [20, 21] and [22]. The second model is an extension of the first for the particular case of $T^{1,1}$, and furnishes an $\mathcal{N} = 4$ supersymmetrization of the PT ansatz [13, 15]. As such, all known conifold solutions can be obtained within it. In this framework, we found a superpotential for the solution describing the baryonic branch of the Klebanov–Strassler theory, which encompasses the previously known superpotentials and reduces to them by taking appropriate limits. We also obtained a superpotential for the solution corresponding to D3 and wrapped D5-branes on the resolved conifold.

For our construction to work and give supersymmetric solutions, it is essential that the constraints in section 3.2 are satisfied. In the concrete examples we studied, the constraints turned out to be compatible with the flow equations. They were solved for some of the scalar fields, which were subsequently integrated out, so that a superpotential and a system of flow equations for a smaller set of fields was obtained. It would be useful to prove that this can be done in more generality.

Regarding technical extensions to our work, it would be interesting to repeat the general analysis by also including the ξ_M components of the $\mathcal{N} = 4$ embedding tensor [26], which were set to zero here (and were vanishing in the particular examples studied). Another obvious development would be to study charged domain walls by allowing a non-trivial profile for some of the $\mathcal{N} = 4$ supergravity form fields. We have restricted our attention to the Poincaré invariant case, but domain walls with curved four-dimensional leaves (mainly AdS) are also of interest (see [48, 49] for the $\mathcal{N} = 2$ case). Finally, the strategy developed in this paper

should be smoothly translatable to the case of four-dimensional, $\mathcal{N} = 4$ gauged supergravity and applicable e.g. to the truncations in [50].

In addition to those discussed in sections 4 and 5, there exist other five-dimensional $\mathcal{N} = 4$ gauged supergravity models that might contain supersymmetric domain wall solutions with a string theory lift. For instance, a simple possibility would be to consider pure supergravity with its inequivalent $SU(2) \times U(1)$ gaugings [51], and study the possible dilatonic domain walls with a non-trivial profile for the scalar in the gravity multiplet, lifted according to [52, 53]. Other appealing examples are the $\mathcal{N} = 4$ supergravities considered in [54] and related to reductions on orbifolds of the sphere.

Finally, it would be interesting to explore the possibility of deforming the superpotential for supersymmetric domain walls to a fake superpotential for non-supersymmetric solutions. It should also be possible to employ the superpotential presented in section 5.1 to study non-supersymmetric deformations to the solution in [7] by using the computational technique presented in [55].

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A More on the scalar manifold geometry

The $SO(5)$ gamma matrices $\Gamma_{\underline{a}i}{}^j$ satisfy

$$\Gamma_{\underline{a}}{}^{ij} = -\Gamma_{\underline{a}}{}^{ji}, \quad \Gamma_{\underline{a}i}{}^i = 0, \quad (\Gamma_{\underline{a}}{}^{ij})^* = \Omega_{ik}\Omega_{jl}\Gamma_{\underline{a}}{}^{kl}, \quad (\text{A.1})$$

as well as

$$\{\Gamma^{\underline{a}}, \Gamma^{\underline{b}}\} = 2\delta^{\underline{ab}}\mathbb{1}_4, \quad (\text{A.2})$$

$$\Gamma^{\underline{a}ij}\Gamma_{\underline{a}kl} = 4\delta_{kl}^{ij} - \Omega^{ij}\Omega_{kl}, \quad (\text{A.3})$$

and the relations between antisymmetric products

$$\Gamma_{\underline{a_1 \dots a_k}} = \frac{(-1)^{[k/2]}}{(5-k)!} \epsilon_{\underline{a_1 \dots a_5}} \Gamma_{\underline{a_{k+1} \dots a_5}} \quad (\text{with } \epsilon_{12345} = +1). \quad (\text{A.4})$$

It is also useful to record that

$$\Gamma_{\underline{ab}ij} \Gamma^{cdij} = 8 \delta_{\underline{ab}}^{cd}, \quad (\text{A.5})$$

so for any antisymmetric tensor $T_{\underline{ab}}$ the relation between its components in vectorial and in spinorial indices is

$$T_{ij} = T_{\underline{cd}} \Gamma^{cd}_{ij} \quad \Leftrightarrow \quad T_{\underline{ab}} = \frac{1}{8} \Gamma_{\underline{ab}ij} T^{ij}. \quad (\text{A.6})$$

The $\frac{\text{SO}(5,n)}{\text{SO}(5) \times \text{SO}(n)}$ coset representative $(\mathcal{V}_M^a, \mathcal{V}_M^a)$ and its inverse $(\mathcal{V}_a^M, \mathcal{V}_a^M)^T$ satisfy

$$\begin{aligned} \mathcal{V}_a^M \mathcal{V}_M^a &= 0 = \mathcal{V}_a^M \mathcal{V}_M^{\underline{a}}, \\ \mathcal{V}_a^M \mathcal{V}_M^b &= \delta_a^b, \\ \mathcal{V}_a^M \mathcal{V}_M^{\underline{b}} &= \delta_a^{\underline{b}}. \end{aligned} \quad (\text{A.7})$$

It follows that from

$$\eta_{MN} = -\mathcal{V}_M^a \mathcal{V}_N^a + \mathcal{V}_M^a \mathcal{V}_N^{\underline{a}} = -\mathcal{V}_M^{ij} \mathcal{V}_N^{ij} + \mathcal{V}_M^a \mathcal{V}_N^{\underline{a}} \quad (\text{A.8})$$

we also have

$$\mathcal{V}_M^a = \eta_{MN} \mathcal{V}^{Na}, \quad \mathcal{V}_M^{\underline{a}} = -\eta_{MN} \mathcal{V}^{N\underline{a}}. \quad (\text{A.9})$$

For the covariant derivative D_x with respect to the $\text{SO}(5) \times \text{SO}(n)$ composite connection on the scalar manifold, one has

$$D_x \mathcal{V}_M^{ij} = -\frac{1}{2} \mathcal{V}_M^a v_x^{a ij}, \quad D_x \mathcal{V}_M^{ij} = +\frac{1}{2} \mathcal{V}_M^{\underline{a}} v_x^{\underline{a} ij}, \quad (\text{A.10})$$

$$D_x \mathcal{V}_M^a = -\frac{1}{2} \mathcal{V}_M^{ij} v_x^a{}_{ij}, \quad D_x \mathcal{V}_M^{\underline{a}} = +\frac{1}{2} \mathcal{V}_M^{ij} v_x^{\underline{a}}{}_{ij}. \quad (\text{A.11})$$

Finally, the curvature two-form of the $\text{SO}(5)$ connection,

$$R^{\underline{ab}} = \frac{1}{4} (d\omega^{\underline{ab}} + \omega^{\underline{a}}{}_{\underline{c}} \wedge \omega^{\underline{cb}}), \quad (\text{A.12})$$

can be computed from (2.4) using (A.10), (2.3), and reads

$$R^{\underline{ab}} = \frac{1}{8} v^{\underline{a}\underline{a}} \wedge v^{\underline{ab}}. \quad (\text{A.13})$$

Introducing $R_{ij} = R^{\underline{ab}} (\Gamma_{\underline{ab}})_{ij}$, this can be written as

$$R_{ij} = \frac{1}{8} v^a{}_{i\ k} \wedge v^a{}_{kj}. \quad (\text{A.14})$$

or, in terms of its components $R_{ij} = \frac{1}{2} R_{xyij} d\phi^x \wedge d\phi^y$, as

$$R_{xyij} = \frac{1}{4} v^a{}_{[x i\ k} v^a{}_{y]kj} = \frac{1}{4} v_x^{\underline{a}\underline{a}} v_y^{\underline{ab}} (\Gamma_{\underline{ab}})_{ij}. \quad (\text{A.15})$$

So we have for the following product of two vielbeine:

$$v_{xi}^a v_{yk}^j = g_{xy} \delta_i^j + 4R_{xyi}^j. \quad (\text{A.16})$$

Moreover, one can check that the following identity holds

$$R_{xyij} R_z^{kl} = \frac{1}{8} v_{x(i}^a v_{zj)}^l - \frac{1}{8} g_{xz} \delta_i^{(k} \delta_j^{l)} - \frac{1}{2} R_{xz(i}^{(k} \delta_j^{l)}. \quad (\text{A.17})$$

By contracting with δ_k^j , this becomes

$$4R_{xyi}^j R_z^{jk} = g_{xz} \delta_i^k + 3R_{xzi}^k. \quad (\text{A.18})$$

B Analysis of the gaugino equation

In this appendix, we analyse the gaugino equation (3.4), namely

$$i \phi^{x'} v_{xij}^a \gamma_5 \epsilon^j + 2g P_i^a{}^j \epsilon_j = 0, \quad (\text{B.1})$$

where the gaugino shift matrix $P_i^a{}^j$ was defined in (2.19). We prove that when W_+ is chosen, and constraint (3.34) on the supersymmetry parameter is taken into account, this gaugino equation is equivalent to the scalar flow equation

$$\phi^{x'} = \mp 3g g^{xy} \partial_y W_+ \quad (\text{B.2})$$

(where the sign choice is the same as the one in (3.34)), together with the algebraic conditions

$$\xi^{aa} X_a = 0, \quad (\text{B.3})$$

$$f^{aab} L_{+ab}^r (\delta^{rs} + Q_+^r Q_+^s) = 0, \quad (\text{B.4})$$

which are just eqs. (3.28), (3.31) and (3.32) in the main text. The same statement holds when W_+ and Q_+ are replaced with W_- and Q_- (we omit the proof since it is completely analogous to the W_+ case).

B.1 Rewriting the equation

As a first thing, we rewrite the gaugino equation in a more convenient way. We introduce the following vectors on the scalar manifold:

$$K^x := -\frac{1}{2} \Sigma^2 v^{xaij} \xi^{MN} \mathcal{V}_M^a \mathcal{V}_{Nij} \quad (\text{B.5})$$

and

$$K_M^x := -\frac{1}{2}\Sigma^{-1}v^{x\,a\,ij}f_M^{NP}\mathcal{V}_N^a\mathcal{V}_{P\,ij}. \quad (\text{B.6})$$

One can show that these vectors are Killing [25], and that the isometries they generate are precisely those being gauged in the supergravity theory specified by the embedding tensor components ξ^{MN} , f^{MNP} . With these definitions, it is easy to see that the gaugino shift matrix P_{ij}^a given in (2.19) can be written as

$$P_{ij}^a = v_{x\,i}^a{}^k \left(-\frac{1}{2\sqrt{2}}\Omega_{kj}K^x - \frac{1}{2}K^{Mx}\mathcal{V}_{M\,kj} \right). \quad (\text{B.7})$$

Then the gaugino equation (B.1) is equivalent to

$$v_{x\,i}^a{}^k \left(i\phi^{x'}\delta_k^j\gamma_5 + \frac{g}{\sqrt{2}}\delta_k^jK^x + gK^{Mx}\mathcal{V}_{M\,k}^j \right) \epsilon_j = 0, \quad (\text{B.8})$$

which is the expression we are going to analyse below.

We also note that, starting from the definitions of the various objects and recalling (A.10), one can prove the following general identity involving the $\text{USp}(4)$ covariant derivative of the gravitino shift matrix, the $\text{USp}(4)$ curvature and the Killing vectors:

$$3D_xP_{ij} = -2\sqrt{2}R_{xy\,ij}K^y - 4R_{xy\,(i}{}^k\mathcal{V}_{M\,k|j)}K^{My}. \quad (\text{B.9})$$

It also holds that

$$K^{My}R_{xy[i}{}^k\mathcal{V}_{M\,k|j]} = \frac{1}{4}K_x^M\mathcal{V}_{M\,ij}, \quad (\text{B.10})$$

which can be used to rewrite the last term in (B.9) in various ways.¹³

B.2 Identities in $\text{su}(2)_\pm$ subspaces

Our proof of the statement given at the beginning of this appendix crucially relies on the $\text{so}(5) \rightarrow \text{so}(4) = \text{su}(2)_+ \oplus \text{su}(2)_-$ decomposition described in section 3.1. So, as a further preliminary step, we now derive some general identities which require no assumptions and just use the $\text{so}(5) \rightarrow \text{so}(4) = \text{su}(2)_+ \oplus \text{su}(2)_-$ decomposition.

¹³Though we will not need this in our proof, let us note that using identities (B.9), (B.10), the gaugino shift matrix can be recast in a more general form. Multiplying (B.9) with a vielbein and using (B.10) gives

$$v^{x\,a}{}_i{}^k \left(D_xP_{kj} + 2\sqrt{2}\Omega_{kj}K_x + 3K_x^M\mathcal{V}_{M\,kj} \right) = 0.$$

Hence, multiplying this with an arbitrary coefficient α , the gaugino shift matrix (B.7) can be rewritten as

$$P_{ij}^a = v^{x\,a}{}_i{}^k \left(\alpha D_xP_{kj} - \frac{3-8\alpha}{6\sqrt{2}}\Omega_{kj}K_x - \frac{1-2\alpha}{2}K_x^M\mathcal{V}_{M\,kj} \right).$$

We first introduce the $\text{su}(2)_\pm$ components of the curvature of the $\text{so}(5)$ connection:

$$(R_\pm^r)_{xy} := \frac{1}{2} R_{xyij} L_\pm^{rij} = 4 R_{xy}{}^{\underline{ab}} (L_\pm^r)_{\underline{ab}} = v_x^{\underline{aa}} v_y^{\underline{ab}} L_\pm^r{}_{\underline{ab}}. \quad (\text{B.11})$$

Contracting identity (A.17) with $L_\pm^{rij} L_\pm^{s}{}_{kl}$, we find that their product satisfies

$$(R_\pm^r)_{xy} (R_\pm^s)_{yz} = -\frac{1}{16} \delta^{rs} (g_{xz} - v_x^{\underline{aa}} v_z^{\underline{ab}} X_{\underline{a}} X_{\underline{b}}) + \frac{1}{4} \epsilon^{rst} (R_\pm^t)_{xz}. \quad (\text{B.12})$$

Note that the first term on the right hand side is a metric constructed with the vielbeine orthogonal to X only.

Another relation that will be useful in our proof below is

$$R_{\pm xy}^r v^{y \underline{aa}} X_{\underline{a}} = 0. \quad (\text{B.13})$$

This is easily seen by using (B.11) to express R_\pm^r in terms of the vielbeine, and noting that in vectorial indices the first equality in (3.21) reads $X^a L_{\pm ab}^r = 0$.

Further useful identities can be deduced starting from the general relation (B.9). Let us contract it with $\frac{1}{2} L_\pm^{rij}$ in order to obtain its $\text{su}(2)_\pm$ components. For the first term on the right hand side we simply use (B.11). For the second term, we notice that

$$L_\pm^{rij} R_{xyi}{}^k \mathcal{V}_{Mkj} = \pm 3 L_\pm^r{}_{[\underline{ab}} X_{\underline{c}]} v_x^{\underline{aa}} v_y^{\underline{ab}} \mathcal{V}_M{}^{\underline{c}}, \quad (\text{B.14})$$

where we used (A.15) and then equation

$$\epsilon_{\underline{abcde}} L_\pm^r{}^{\underline{cd}} X_{\underline{e}} = \mp 2 L_\pm^r{}_{\underline{ab}}, \quad (\text{B.15})$$

which is just the second equality in (3.21) expressed in vectorial indices.

By further massaging the different terms arising from the antisymmetrization, we eventually arrive at

$$3 D_x P_\pm^r = -2\sqrt{2} R_{\pm xy}^r (K^y \pm \sqrt{2} K^{My} \mathcal{V}_M{}^a X_{\underline{a}}) \pm 2 \Sigma^{-1} v_x^{\underline{ac}} X_{\underline{c}} L_\pm^r{}_{\underline{ab}} f^{\underline{aab}}, \quad (\text{B.16})$$

where D_x is now an $\text{SU}(2)$ covariant derivative: $DP_\pm^r = dP_\pm^r + \frac{1}{2} \epsilon^{rst} \omega_\pm^s P_\pm^t$, with ω_\pm^s the $\text{SU}(2)_\pm$ components of the $\text{SO}(5)$ connection.¹⁴ Substituting $P_\pm^r = -i W_\pm Q_\pm^r$ gives

$$3i \partial_x W_\pm Q_\pm^r + 3i W_\pm D_x Q_\pm^r = 2\sqrt{2} R_{\pm xy}^r (K^y \pm \sqrt{2} K^{My} \mathcal{V}_M{}^a X_{\underline{a}}) \mp 2 \Sigma^{-1} v_x^{\underline{ac}} X_{\underline{c}} L_\pm^r{}_{\underline{ab}} f^{\underline{aab}}. \quad (\text{B.17})$$

¹⁴A remark is in order about the $\text{su}(2)$ components of DP . Writing P_{ij} as in (3.25), one finds that $\frac{1}{2} L_\pm^{rij} D_x P_{ij} = D_x P_\pm^r$ plus extra terms proportional to $D_x L_\pm^r$ or $D_x L_\pm^r$. By working out the covariant derivative D_x of the generators L , one sees that in this expression it reduces to a partial derivative, ∂_x . Then, working in an $\text{SO}(5)$ gauge in which X is constant, so that the same holds for the L 's (recall footnote 3), we obtain $\frac{1}{2} L_\pm^{rij} D_x P_{ij} = D_x P_\pm^r$, which is the result displayed in (B.16). Anyway, we also cross-checked that our results remain the same when no $\text{SO}(5)$ gauge-fixing is done and the extra terms are included.

We can decompose this equation in the component parallel to Q_\pm^r and its orthogonal part. The former is obtained by contracting with Q_\pm^r . Noting that from (3.17), (3.22) it follows that

$$Q_\pm^r Q_\pm^r = -1 \quad (\text{B.18})$$

and therefore that

$$Q_\pm^r D_x Q_\pm^r = 0, \quad (\text{B.19})$$

we obtain

$$-3i \partial_x W_\pm = 2\sqrt{2} Q_\pm^r R_{\pm xy}^r (K^y \pm \sqrt{2} K^{My} \mathcal{V}_M^a X_a) \mp 2\Sigma^{-1} v_x^{ac} X_c Q_\pm^r L_{\pm ab}^r f^{ab}. \quad (\text{B.20})$$

The orthogonal components are obtained by projecting (B.17) with $(\delta^{rs} + Q_\pm^r Q_\pm^s)$, which gives

$$3i W_\pm D_x Q_\pm^r = (\delta^{rs} + Q_\pm^r Q_\pm^s) \left[2\sqrt{2} R_{\pm xy}^s (K^y \pm \sqrt{2} K^{My} \mathcal{V}_M^a X_a) \mp 2\Sigma^{-1} v_x^{ac} X_c L_{\pm ab}^s f^{ab} \right]. \quad (\text{B.21})$$

Finally, we multiply (B.20) by $Q_\pm^s R_\pm^s$. Using (B.12) for the first term on the right hand side, noting that $v_z^{ab} X_b K^{zM} \mathcal{V}_M^a X_a = 0$ identically, and recalling (B.13) to see that the last term on the right hand side does not contribute, we arrive at

$$(g_{xz} - v_x^{aa} X_a v_y^{ab} X_b) K^y \pm \sqrt{2} K_x^M \mathcal{V}_M^c X_c = -12\sqrt{2} i Q_\pm^r R_{\pm xy}^r \partial_y W_\pm. \quad (\text{B.22})$$

Given this set of identities, we are now in the position of proving our statement.

B.3 Necessity

We now come to our actual proof. We first show that conditions (B.2), (B.3) and (B.4) follow from the gaugino equation (B.8), i.e. are necessary conditions. We start contracting (B.8) with a vielbein:

$$v_{xi}^a v_{yl}^l v_{jk}^k \left(i \phi^{y'} \delta_k^j \gamma_5 + \frac{g}{\sqrt{2}} \delta_k^j K^y + g K^{My} \mathcal{V}_{Mk}^j \right) \epsilon_j = 0. \quad (\text{B.23})$$

Then we use (A.16) to express the product of the two vielbeine, and constraint (3.34) to eliminate γ_5 . Moreover, we invoke (B.9), (B.10) to eliminate the terms that involve both the curvature R and the Killing vectors K, K_M . In this way we arrive at

$$\pm i \phi^{y'} (g_{xy} \delta_i^k + 4R_{xy} i^k) Q_{+k}^j \epsilon_j - 3g D_x P_i^j \epsilon_j + \frac{g}{\sqrt{2}} K_x \epsilon_i + 2g K_x^M \mathcal{V}_{Mi}^j \epsilon_j = 0, \quad (\text{B.24})$$

where the sign choice in the first term is inherited from (3.34). By projecting with Π_+ and massaging the different terms (also recalling $\Pi_- \epsilon = 0$), we obtain an equation of the form

$(\alpha \delta_i^j + \beta^r L_{+i}^r) \epsilon_j = 0$. This is equivalent to $\alpha = \beta^r = 0$, as it can be seen by applying $(\alpha \delta_i^j + \beta^r L_{+i}^r)$ another time and using property (3.22) of the $\mathfrak{su}(2)$ generators. In our case, $\alpha = 0$ corresponds to

$$K_x + \sqrt{2} K_x^M \mathcal{V}_M^a X_a = \pm 4\sqrt{2} i g^{-1} Q_+^r R_{+xy}^r \phi^{y'}, \quad (\text{B.25})$$

while $\beta^r = 0$ is

$$3gi D_x P_+^r = \mp g_{xy} \phi^{y'} Q_+^r \pm 4 \epsilon^{rst} Q_+^s R_{+xy}^t \phi^{y'}. \quad (\text{B.26})$$

Since $P_\pm^r = -iW_\pm Q_\pm^r$, the latter is the same as

$$3g \partial_x W_+ Q_+^r + 3g W_+ D_x Q_+^r = \mp g_{xy} \phi^{y'} Q_+^r \pm 4 \epsilon^{rst} Q_+^s R_{+xy}^t \phi^{y'}. \quad (\text{B.27})$$

Now, eq. (B.25) leads to our condition (B.3): this is obtained multiplying (B.25) by $X_a v_x^{aa}$, recalling (B.13), noting that $K^{Mx} \mathcal{V}_M^b X_b v_x^{aa} X_a = 0$ identically and then expanding K^x .

On the other hand, recalling (B.18), (B.19), we see that the component of (B.27) along Q_+^r is just the wanted flow equation (B.2). Finally, the part of (B.27) orthogonal to Q_+^r is

$$3g W_+ D_x Q_+^r = \pm 4 \epsilon^{rst} Q_+^s R_{+xy}^t \phi^{y'}, \quad (\text{B.28})$$

and leads to our constraint (B.4). This is seen by contracting the general identity (B.21) with $v^{xad} X_d$: the term in (B.21) involving R_+^s vanishes by (B.13), and the same happens for the term involving $D_x Q_+$, once (B.28) is invoked.

As an aside, note that (B.28) also implies $\phi^{x'} D_x Q_+^r = 0$, which is one of the constraints we found while analysing the gravitino equation in section 3.3. Therefore this constraint is automatically satisfied once the gaugino equation is solved.

B.4 Sufficiency

It remains to prove that the flow equation (B.2) together with constraints (B.3), (B.4) are sufficient conditions for the gaugino equation (B.8). To see this, we first show that they imply (B.25), as well as the following relation involving the component of the $\mathfrak{so}(5)$ -valued matrix f^{aab} parallel to Q_+^r :

$$\mp i \phi^{x'} v_x^{aa} X_a = 2g \Sigma^{-1} Q_+^r L_{+ab}^r f^{aab}. \quad (\text{B.29})$$

Eq. (B.25) is immediately obtained from identity (B.22) by using (B.2), (B.3). On the other hand, contracting identity (B.20) with $X_a v^{xaa}$ and recalling (B.13) yields

$$3i \partial_x W_+ v^{xaa} X_a = 2 \Sigma^{-1} Q_+^r L_{+ab}^r f^{aab}, \quad (\text{B.30})$$

which by using (B.2) becomes the wanted relation (B.29).

We next show that, together with constraints (3.34) and (B.4), the relations just obtained satisfy the gaugino equation (B.8). Using constraint (3.34) to eliminate γ_5 , as well as (B.25) to eliminate K^x , eq. (B.8) becomes

$$\mp i \phi^{x'} \left(v_{xi}^a{}^k Q_{+k}^j + 4 v^{ya}{}_i{}^j R_{+yx}^r Q_+^r \right) \epsilon_j + g K^{Mx} v_{xi}^a{}^k \left(\delta_k^j \mathcal{V}_M^a X_{\underline{a}} - \mathcal{V}_{Mk}^j \right) \epsilon_j = 0. \quad (\text{B.31})$$

Let us elaborate the first and the second term separately. Expressing R_+^r as in (B.11), one can check that

$$v_{xi}^a{}^k Q_{+k}^j + 4 v^{ya}{}_i{}^j R_{+yx}^r Q_+^r = Q_{+i}^k v_{xk}^a{}^j. \quad (\text{B.32})$$

Also noting that

$$Q_+ v_x^a = Q_+ \Pi_+ v_x^a = v_x^{aa} X_{\underline{a}} Q_+ + Q_+ v_x^a \Pi_-, \quad (\text{B.33})$$

and that the last term annihilates ϵ , we obtain for the first term of (B.31):

$$\begin{aligned} \mp i \phi^{x'} \left(v_{xi}^a{}^k Q_{+k}^j + 4 v^{ya}{}_i{}^j R_{+yx}^r Q_+^r \right) \epsilon_j &= \mp i \phi^{x'} v_x^{aa} X_{\underline{a}} Q_{+i}^j \epsilon_j \\ &= 2g \Sigma^{-1} Q_+^r L_{+ab}^r f^{aab} Q_{+i}^j \epsilon_j, \end{aligned} \quad (\text{B.34})$$

where in the last equality we used condition (B.29). On the other hand, the second term of (B.31) can be rewritten as

$$\begin{aligned} g K^{Mx} v_{xi}^a{}^k \left(\delta_k^j \mathcal{V}_M^a X_{\underline{a}} - \mathcal{V}_{Mk}^j \right) \epsilon_j &= g K^{Mx} v_{xi}^a{}^k \left[X_k^l \mathcal{V}_{Ml}^j - 2 \mathcal{V}_{Mk}^l \Pi_{-l}^j \right] \epsilon_j \\ &= g K^{Mx} v_{xi}^a{}^k X_k^l \mathcal{V}_{Ml}^j \epsilon_j \\ &= -\frac{1}{4} g \Sigma^{-1} f^{aab} \epsilon_{\underline{abcde}} X^e \Gamma_{\underline{cd}}^{\underline{cd}}{}_i{}^j \epsilon_j, \end{aligned} \quad (\text{B.35})$$

where the last equality is obtained by expanding K^{Mx} . So (B.31) becomes

$$f^{aab} \left(8 Q_+^r L_{+ab}^r Q_{+i}^j - \epsilon_{\underline{abcde}} X^e \Gamma_{\underline{cd}}^{\underline{cd}}{}_i{}^j \right) \epsilon_j = 0. \quad (\text{B.36})$$

Let us elaborate the second term in parenthesis. We can write

$$\begin{aligned} \epsilon_{\underline{abcde}} X^e \Gamma_{\underline{cd}}^{\underline{cd}}{}_i{}^j \epsilon_j &= \epsilon_{\underline{abcde}} X^e \left(\Gamma_{\underline{cd}}^{\underline{cd}} \Pi_+ \Pi_+ \right)_i{}^j \epsilon_j \\ &= \epsilon_{\underline{abcde}} X^e \left(\Pi_+ \Gamma_{\underline{cd}}^{\underline{cd}} \Pi_+ \right)_i{}^j \epsilon_j \\ &= 4 \epsilon_{\underline{abcde}} X^e L_{+}^r{}^{\underline{cd}} L_{+i}^r{}^j \epsilon_j \\ &= -8 L_{+ab}^r L_{+i}^r{}^j \epsilon_j, \end{aligned} \quad (\text{B.37})$$

where in the first equality we used $\epsilon = \Pi_+ \epsilon$ twice, in the second we noted that $X \equiv X_{\underline{f}} \Gamma^{\underline{f}}$ commutes with $\epsilon_{\underline{abcde}} X^e \Gamma^{\underline{cd}}$, in the third we used the completeness relation (3.24), and in the last we recalled (B.15). So (B.36) becomes

$$f^{a\underline{ab}} L_{+a\underline{b}}^r (\delta^{rs} + Q_+^r Q_+^s) L_{+i}^s \epsilon_j = 0, \quad (\text{B.38})$$

which is satisfied by (B.4).

This concludes our proof that the gaugino equation (B.8) is equivalent to (B.2), (B.3) and (B.4) once the constraint (3.34) on the spinor is taken into account and the choice of W_+ is made.

References

- [1] L. Girardello, M. Petrini, M. Porrati and A. Zaffaroni, *Novel local CFT and exact results on perturbations of $N=4$ superYang Mills from AdS dynamics*, JHEP **9812** (1998) 022 [hep-th/9810126].
- [2] D. Z. Freedman, S. S. Gubser, K. Pilch and N. P. Warner, *Renormalization group flows from holography supersymmetry and a c theorem*, Adv. Theor. Math. Phys. **3** (1999) 363 [hep-th/9904017].
- [3] I. R. Klebanov and A. A. Tseytlin, *Gravity Duals of Supersymmetric $SU(N) \times SU(N+M)$ Gauge Theories*, Nucl. Phys. B **578** (2000) 123 [arXiv:hep-th/0002159].
- [4] I. R. Klebanov and M. J. Strassler, *Supergravity and a confining gauge theory: Duality cascades and $chiSB$ -resolution of naked singularities*, JHEP **0008** (2000) 052 [arXiv:hep-th/0007191].
- [5] J. M. Maldacena and C. Nunez, *Towards the large N limit of pure $N = 1$ super Yang Mills*, Phys. Rev. Lett. **86**, 588 (2001) [arXiv:hep-th/0008001].
- [6] L. A. Pando Zayas and A. A. Tseytlin, *3-branes on resolved conifold*, JHEP **0011** (2000) 028 [arXiv:hep-th/0010088].
- [7] A. Butti, M. Grana, R. Minasian, M. Petrini and A. Zaffaroni, *The baryonic branch of Klebanov-Strassler solution: A supersymmetric family of $SU(3)$ structure backgrounds*, JHEP **0503**, 069 (2005) [arXiv:hep-th/0412187].
- [8] K. Skenderis and P. K. Townsend, *Gravitational stability and renormalization group flow*, Phys. Lett. B **468** (1999) 46 [hep-th/9909070].

- [9] M. Cvetič and H. H. Soleng, *Supergravity domain walls*, Phys. Rept. **282** (1997) 159 [hep-th/9604090].
- [10] K. Behrndt and M. Cvetič, *Supersymmetric domain wall world from $D = 5$ simple gauged supergravity*, Phys. Lett. B **475** (2000) 253 [hep-th/9909058].
- [11] A. Ceresole, G. Dall'Agata, R. Kallosh and A. Van Proeyen, *Hypermultiplets, domain walls and supersymmetric attractors*, Phys. Rev. D **64** (2001) 104006 [hep-th/0104056].
- [12] N. Halmagyi, J. T. Liu and P. Szepietowski, *On $N = 2$ Truncations of IIB on $T^{1,1}$* , JHEP **1207** (2012) 098 [arXiv:1111.6567 [hep-th]].
- [13] D. Cassani and A. F. Faedo, *A Supersymmetric consistent truncation for conifold solutions*, Nucl. Phys. B **843** (2011) 455 [arXiv:1008.0883 [hep-th]].
- [14] S. S. Gubser, C. P. Herzog and I. R. Klebanov, *Symmetry breaking and axionic strings in the warped deformed conifold*, JHEP **0409** (2004) 036 [hep-th/0405282].
- [15] I. Bena, G. Giecold, M. Grana, N. Halmagyi and F. Orsi, *Supersymmetric Consistent Truncations of IIB on $T^{1,1}$* , JHEP **1104** (2011) 021 [arXiv:1008.0983 [hep-th]].
- [16] G. Giecold, *Remark on the Baryonic Branch of the Warped Deformed Conifold*, JHEP **1206** (2012) 035 [arXiv:1112.1054 [hep-th]].
- [17] M. Zagermann, *$N=4$ fake supergravity*, Phys. Rev. D **71** (2005) 125007 [hep-th/0412081].
- [18] D. Cassani, G. Dall'Agata and A. F. Faedo, *Type IIB supergravity on squashed Sasaki-Einstein manifolds*, JHEP **1005** (2010) 094 [arXiv:1003.4283 [hep-th]].
- [19] J. P. Gauntlett and O. Varela, *Universal Kaluza-Klein reductions of type IIB to $N = 4$ supergravity in five dimensions*, JHEP **1006** (2010) 081 [arXiv:1003.5642 [hep-th]].
- [20] L. A. Pando Zayas and A. A. Tseytlin, *3-Branes on Spaces with $R \times S^2 \times S^3$ Topology*, Phys. Rev. D **63** (2001) 086006 [hep-th/0101043].
- [21] S. Benvenuti, M. Mahato, L. A. Pando Zayas and Y. Tachikawa, *The Gauge/gravity theory of blown up four cycles*, hep-th/0512061.
- [22] L. Girardello, M. Petrini, M. Porrati and A. Zaffaroni, *The Supergravity dual of $N=1$ superYang-Mills theory*, Nucl. Phys. B **569** (2000) 451 [hep-th/9909047].
- [23] G. Papadopoulos and A. A. Tseytlin, *Complex geometry of conifolds and five-brane wrapped on two sphere*, Class. Quant. Grav. **18** (2001) 1333 [hep-th/0012034].

- [24] M. Awada and P. K. Townsend, *N=4 Maxwell-Einstein Supergravity In Five-dimensions And Its SU(2) Gauging*, Nucl. Phys. B **255** (1985) 617.
- [25] G. Dall'Agata, C. Herrmann and M. Zagermann, *General matter coupled N=4 gauged supergravity in five-dimensions*, Nucl. Phys. B **612** (2001) 123 [hep-th/0103106].
- [26] J. Schön and M. Weidner, *Gauged N = 4 supergravities*, JHEP **0605** (2006) 034 [arXiv:hep-th/0602024].
- [27] H. Samtleben, *Lectures on Gauged Supergravity and Flux Compactifications*, Class. Quant. Grav. **25** (2008) 214002 [arXiv:0808.4076 [hep-th]].
- [28] J. T. Liu, P. Szepietowski and Z. Zhao, *Consistent massive truncations of IIB supergravity on Sasaki-Einstein manifolds*, Phys. Rev. D **81** (2010) 124028 [arXiv:1003.5374 [hep-th]].
- [29] K. Skenderis, M. Taylor and D. Tsimpis, *A Consistent truncation of IIB supergravity on manifolds admitting a Sasaki-Einstein structure*, JHEP **1006** (2010) 025 [arXiv:1003.5657 [hep-th]].
- [30] J. T. Liu, P. Szepietowski and Z. Zhao, *Supersymmetric massive truncations of IIB supergravity on Sasaki-Einstein manifolds*, Phys. Rev. D **82** (2010) 124022 [arXiv:1009.4210 [hep-th]].
- [31] I. Bah, A. Faraggi, J. I. Jottar and R. G. Leigh, *Fermions and Type IIB Supergravity On Squashed Sasaki-Einstein Manifolds*, JHEP **1101** (2011) 100 [arXiv:1009.1615 [hep-th]].
- [32] F. Benini, F. Canoura, S. Cremonesi, C. Nunez and A. V. Ramallo, *Unquenched flavors in the Klebanov-Witten model*, JHEP **0702** (2007) 090 [hep-th/0612118].
- [33] M. Grana and J. Polchinski, *Supersymmetric three form flux perturbations on AdS(5)*, Phys. Rev. D **63** (2001) 026001 [hep-th/0009211].
- [34] K. Pilch and N. P. Warner, *N=1 supersymmetric renormalization group flows from IIB supergravity*, Adv. Theor. Math. Phys. **4** (2002) 627 [hep-th/0006066].
- [35] P. Candelas and X. C. de la Ossa, *Comments on Conifolds*, Nucl. Phys. B **342** (1990) 246.
- [36] I. R. Klebanov and E. Witten, *Superconformal field theory on three-branes at a Calabi-Yau singularity*, Nucl. Phys. B **536** (1998) 199 [hep-th/9807080].

- [37] I. R. Klebanov and N. A. Nekrasov, *Gravity duals of fractional branes and logarithmic RG flow*, Nucl. Phys. B **574** (2000) 263 [hep-th/9911096].
- [38] J. T. Liu and P. Szepietowski, *Supersymmetry of consistent massive truncations of IIB supergravity*, Phys. Rev. D **85** (2012) 126010 [arXiv:1103.0029 [hep-th]].
- [39] M. Berg, M. Haack and W. Mueck, *Bulk dynamics in confining gauge theories*, Nucl. Phys. B **736** (2006) 82 [hep-th/0507285].
- [40] R. Casero, C. Nunez and A. Paredes, *Towards the string dual of $N=1$ SQCD-like theories*, Phys. Rev. D **73** (2006) 086005 [hep-th/0602027].
- [41] C. Hoyos-Badajoz, C. Nunez and I. Papadimitriou, *Comments on the String dual to $N=1$ SQCD*, Phys. Rev. D **78** (2008) 086005 [arXiv:0807.3039 [hep-th]].
- [42] M. Cvetič, G. W. Gibbons, H. Lu and C. N. Pope, *Ricci flat metrics, harmonic forms and brane resolutions*, Commun. Math. Phys. **232** (2003) 457 [arXiv:hep-th/0012011].
- [43] R. Gwyn and A. Knauf, *Conifolds and geometric transitions*, Rev. Mod. Phys. **80** (2008) 1419 [arXiv:hep-th/0703289].
- [44] J. Maldacena, D. Martelli, *The Unwarped, resolved, deformed conifold: Fivebranes and the baryonic branch of the Klebanov-Strassler theory*, JHEP **1001** (2010) 104. [arXiv:0906.0591 [hep-th]].
- [45] D. Elander, J. Gaillard, C. Nunez and M. Piai, *Towards multi-scale dynamics on the baryonic branch of Klebanov-Strassler*, JHEP **1107** (2011) 056 [arXiv:1104.3963 [hep-th]].
- [46] R. C. Myers, *Dielectric branes*, JHEP **9912** (1999) 022 [arXiv:hep-th/9910053].
- [47] K. Pilch and N. P. Warner, *$N = 1$ supersymmetric solutions of IIB supergravity from Killing spinors*, arXiv:hep-th/0403005.
- [48] G. Lopes Cardoso, G. Dall'Agata and D. Lust, *Curved BPS domain wall solutions in five-dimensional gauged supergravity*, JHEP **0107** (2001) 026 [hep-th/0104156].
- [49] G. Lopes Cardoso, G. Dall'Agata and D. Lust, *Curved BPS domain walls and RG flow in five-dimensions*, JHEP **0203** (2002) 044 [hep-th/0201270].
- [50] D. Cassani and P. Koerber, *Tri-Sasakian consistent reduction*, JHEP **1201** (2012) 086 [arXiv:1110.5327 [hep-th]].

- [51] L. J. Romans, *Gauged $N=4$ Supergravities In Five-dimensions And Their Magnetovac Backgrounds,* *Nucl. Phys. B* **267** (1986) 433.
- [52] H. Lu, C. N. Pope and T. A. Tran, *Five-dimensional $N=4$, $SU(2) \times U(1)$ gauged supergravity from type IIB,* *Phys. Lett. B* **475** (2000) 261 [hep-th/9909203].
- [53] J. P. Gauntlett and O. Varela, *$D=5$ $SU(2) \times U(1)$ Gauged Supergravity from $D=11$ Supergravity,* *JHEP* **0802** (2008) 083 [arXiv:0712.3560 [hep-th]].
- [54] R. Corrado, M. Gunaydin, N. P. Warner and M. Zagermann, *Orbifolds and flows from gauged supergravity,* *Phys. Rev. D* **65** (2002) 125024 [hep-th/0203057].
- [55] V. Borokhov and S. S. Gubser, *Nonsupersymmetric deformations of the dual of a confining gauge theory,* *JHEP* **0305** (2003) 034 [hep-th/0206098].